6.1 ON THE SPECTRAL DENSITY OF SOME STOCHASTIC PROCESSES

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1. Introduction.

We prove the following theorem, which was motivated by a question that Wyner raised in [1].

Theorem: Given any $\varepsilon > 0$ and $A > 0$, there is a complex stationary stochastic process $x(t, w)$ which satisfies:

(i) $|x(t, w)| \le A$ a.s.

(ii) $\| S_x(f) - B_A(f) \|_1 \le \varepsilon$,

where $S_x(f) = \int e^{-2\pi i f \tau} E\, x(t)\, \overline{x(t+\tau)}\, d\tau$ is the spectral density of x and

$$B_A(f) = \begin{cases} A^2/2 & |f| \le 1 \\ 0 & |f| > 1 \end{cases}$$

is the boxcar spectral density with bandwidth 1 and total power A^2.

In fact, we have (ii) from the following stronger set of conclusions:

(iii) $S_x(f) \ge 0$ and S_x is even.

(iv) $\int_1^{\infty} S_x(f)\, df < \varepsilon$ and $| \int_{-1}^{1} S_x(f) - A^2 | < \varepsilon$.

(v) $| \max_{|f| \le 1} S_x(f) - A^2/2 | < \varepsilon$.

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Thus x is a process with nearly boxcar spectrum which is not only power limited to A^2 but is amplitude limited to A (a stricter constraint). Moreover, the process we construct is *ergodic*. Aaron Wyner has pointed out to us that there are quite simple constructions of processes satisfying (i) and (ii) above, but they are not ergodic. The construction of our process is more delicate and thus the verification of the properties of the process is at least as interesting as the properties themselves.

We also have the following corollary whose proof is immediate:

Corollary: The process x above satisfies:

$$\int_{-1}^{1} \log(1 + S_x(f))\, df \geq 2 \log(1 + \frac{A^2}{2}) - \varepsilon$$

$$= \int_{-1}^{1} \log(1 + B_A(f))\, df - \varepsilon .$$

2. Proof of the Theorem.

We now prove the theorem.

Proof. In [2], p. 321, J.P. Kahane demonstrates that there are polynomials,

$$P_n(z) = \sum_{m=1}^{n} a_{mn} z^n, \quad |a_{mn}| = 1 ,$$

and $\varepsilon_n \to 0$ such that

$$\| P_n(e^{i\theta}) \|_\infty \leq (1 + \varepsilon_n) \sqrt{n} .$$

In fact, he even proves a stronger result, but we shall not need this. Let

$$u_n(t) = \frac{A}{\sqrt{2N+1}} e^{-2\pi i t/N} P_{2N+1}(e^{2\pi i t/N}) .$$

u_n is a N periodic signal with power A^2 and peak

$$\| u_n \|_\infty \leq (1 + \varepsilon_n) A .$$

Let

$$U_N(t, \omega) = u_N(t + \theta(\omega)) ,$$

where $\theta(\omega)$ is uniformly distributed on $[0, N]$. U_N is a complex stationary stochastic process such that

$$\| U_N \|_\infty \leq (1 + \varepsilon_N) A \quad \text{a.s.}$$

and with spectral measure

$$S_{U_N}(f) = \frac{A^2}{2N+1} \sum_{|n| \leq N} \delta(f - \frac{n}{N}).$$

These spectral measures approximate the boxcar spectrum in distribution but we want a stronger approximation of the densities.

To do this, we modulate the process U_N as follows: Let $Z_{N,\alpha}$ be random telegraph process with rate $\alpha/2\pi N$, independent of U_N, where $\alpha > 1$. Then,

$$| Z_{N,\alpha} | = 1 \quad \text{a.s.}$$

and

$$S_{Z_{N,\alpha}}(f) = \frac{\alpha \pi^{-1} N}{\alpha^2 + (Nf)^2}.$$

Let

$$X_{N,\alpha} = \frac{Z_{N,\alpha} U_N}{1 + \varepsilon_N}.$$

Then

$$| X_{N,\alpha} | \leq A \text{ a.s}$$

and

$$S_{X_{N,\alpha}} = \frac{1}{(1+\varepsilon_N)^2} \frac{2N}{2N+1} \frac{A^2}{2\pi} \sum_{|n| \leq N} \frac{\alpha}{\alpha^2 + (Nf + n)^2} .$$

The theorem now follows at once from the lemmas below by choosing N and α large enough. (See Lemma F in particular.) □

Lemma A: For fixed $\alpha > 1$,

$$\overline{\lim_{N\to\infty}} \| S_{X_{N,\alpha}} - B_A \|_1 \le 4 \overline{\lim_{N\to\infty}} \max_{f \in [-1,1]} \left| | S_{X_{N,\alpha}} | - \frac{A^2}{2} \right|$$

Proof. Note that $S_{X_{N,\alpha}}(f)$ is an even function. We show first that

$$\int_1^\infty S_{X_{N,\alpha}}(f)\, df \to 0 .$$

Now

$$\int_1^\infty \frac{\alpha}{\alpha^2 + (Nf-n)^2}\, df = \frac{1}{N}\left[\frac{\pi}{2} - \tan^{-1}\left[N - \frac{n}{\alpha}\right]\right]$$

and so

$$\int_1^\infty \sum_{|n| \le N} \frac{\alpha}{\alpha^2 + (Nf-n)^2}\, df = \frac{2}{2N} \sum_{n=0}^{2N} \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{n}{\alpha}\right)\right] \to 0$$

by Cesaro convergence. Therefore,

$$\int_1^\infty S_{X_{N,\alpha}}(f) df \to 0 .$$

Similarly, since $S_{X_{N,\alpha}}(f)$ is even,

$$\int_{-\infty}^{-1} S_{X_{N,\alpha}}(f) df \to 0 .$$

Also, by a similar calculation,

$$\int_{-\infty}^{\infty} S_{X_{N,\alpha}}(f)\, df = \frac{1}{(1+\varepsilon_N)^2} A^2 .$$

Now

$$\| S_{X_{N,\alpha}} - B_A \|_1 = \int_{-1}^{1} \left| S_{X_{N,\alpha}} - \frac{A^2}{2} \right| df + \int_1^\infty S_{X_{N,\alpha}}(f) df + \int_{-\infty}^{-1} S_{X_{N,\alpha}}(f) df$$

and so

$$\overline{\lim} \| S_{X_{N,\alpha}} - B_A \|_1 \le \overline{\lim} \int_{-1}^{1} \left| S_{X_{N,\alpha}} - \frac{A^2}{2} \right| df .$$

Now

$$\int_{-1}^{1} \left| S_{X_{N,\alpha}} - \frac{A^2}{2} \right| df = \int_{[S_{X_{N,\alpha}} \ge \frac{A^2}{2}] \cap [-1,1]} (S_{X_{N,\alpha}} - \frac{A^2}{2}) df + \int_{[S_{X_{N,\alpha}} \le \frac{A^2}{2}] \cap [-1,1]} (\frac{A^2}{2} - S_{X_{N,\alpha}}) df$$

$$\int_{S_{X_{N,\alpha}} \ge \frac{A^2}{2}} S_{X_{N,\alpha}} - \frac{A^2}{2}\, df \le \left| \| S_{X_{N,\alpha}} \|_\infty - \frac{A^2}{2} \right| \lambda \left[f \,\middle|\, |f| \le 1,\, S_{X_{N,\alpha}} \ge \frac{A^2}{2} \right]$$

$$\le 2 \left| \| S_{X_{N,\alpha}} \|_\infty - \frac{A^2}{2} \right| ,$$

where $\| S_{X_{N,\alpha}} \|_\infty = \max_{f \in [-1,1]} | S_{X_{N,\alpha}} |$. Moreover,

$$\int_{[S_{X_{N,\alpha}} \le \frac{A^2}{2}] \cap [-1,1]} \left[\frac{A^2}{2} - S_{X_{N,\alpha}}\right] df = A^2 - \int_{-1}^{1} S_{X_{N,\alpha}}\, df + \int_{S_{X_{N,\alpha}} > \frac{A^2}{2}} \left[S_{X_{N,\alpha}} - \frac{A^2}{2}\right] df .$$

Therefore, $\overline{\lim} \| S_{X_{N,\alpha}} - B_N \|_1 \le 4\, \overline{\lim} \left| \| S_{X_{N,\alpha}} \|_\infty - \frac{A^2}{2} \right|$. □

Lemma B: Let $f(x) = \sum_{|n| \le N} \frac{\alpha}{\alpha^2 + (x-n)^2}$. Then $\max_{x \in [-N,N]} f(x) = \max_{x \in [-1,1]} f(x)$.

Proof. Since $f(x)$ is even, it suffices to show $\max_{x \in [0,N]} f(x) = \max_{x \in [0,1]} f(x)$. Fix $y \in [0,1]$ and let $s_k = f(y+k)$ for $k = 0, 1, \ldots, N-1$. We show $s_0 \ge s_1 \ge s_2 \ge \cdots \ge s_{N-1}$. This clearly suffices to finish the proof. Now

$$s_k - s_{k+1} = \sum_{j=k-N}^{k+N} \frac{\alpha}{\alpha + (y+j)^2} - \sum_{j=k+1}^{k+1+N} \frac{\alpha}{\alpha^2 + (y+j)^2}$$

$$= \frac{\alpha}{\alpha^2 + (y+k-N)^2} - \frac{\alpha}{\alpha^2 + (y+k+1+N)^2} \ge 0 . \quad □$$

Lemma C: Let C_N be a square with vertices at $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$, and $(N+\frac{1}{2})(1-i)$. Let $g(z)$ be a function with poles at $z = p_1, \ldots, p_k$ (and assume N is large enough so the C_N contains all these poles within its interior). Suppose that $|g(z)| = O\left(\frac{1}{|z|^2}\right)$ on C_N. Then

$$\sum_{n=-N}^{N} g(n) = \left[-\sum_{j=1}^{k} \text{Residue } (\pi \cot \pi z g(z) \text{ at } p_j)\right] + O(\frac{1}{N})$$

This is a standard fact from the theory of residues.

Lemma D: For $a, b, c, d \in \mathbf{R}$ with $a \neq 0$ we have,

$$\sum_{n=-N}^{N} \frac{d}{(an+b)^2 + c^2} = \frac{\pi d}{2i\mu a^2} (\cot w - \cot \overline{w})$$

where $w = \pi(-\lambda i - \mu)$ and $\lambda = \frac{b}{a}$, $\mu = \frac{c}{a}$. Lemma D follows at once from Lemma C after calculating residues and elementary algebra.

Lemma E:

$$\max_{f \in [-1,1]} |S_{X_{N,\alpha}}|$$

$$= \frac{1}{(1+\varepsilon_N)^2} \frac{2N}{2N+1} \frac{A^2}{2} \max_{x \in [0,1]} \frac{\sec^2 \pi x \coth \pi\alpha}{1 + \coth^2 \pi\alpha \tan^2 \pi x} + O(\frac{1}{N}).$$

Proof.

$$\| S_{X_{N,\alpha}} \|_\infty = \max_{f \in [-1,1]} |S_{X_{N,\alpha}}|$$

$$= \max_{f \in [-1,1]} S_{X_{N,\alpha}} \quad \text{(since } S_{X_{N,\alpha}} \text{ is even and positive)}$$

$$= \frac{1}{(1+\varepsilon_N)^2} \frac{2N}{2N+1} \frac{A^2}{2\pi} \max_{f \in [0,N]} \sum_{|n| \le N} \frac{\alpha}{\alpha^2 + (Nf - n)^2}$$

$$= \frac{1}{(1+\varepsilon_N)^2} \frac{2N}{2N+1} \frac{A^2}{2\pi} \max_{x \in [0,N]} f(x),$$

where $f(x) = \sum_{|n| \le N} \frac{\alpha}{\alpha^2 + (x-n)^2}$. By Lemma B, $\max_{x \in [0,N]} f(x) = \max_{x \in [0,1]} f(x)$. Setting $d = \alpha$, $a = 1$, $c = \alpha$, and $b = -x$ in Lemma D gives that

$$f(x) = \frac{\pi}{2i} (\cot \pi (x - i\alpha) - \cot \pi (x + i\alpha)) + O(\frac{1}{N})$$

$$= \pi(\frac{\sec^2 \pi x \coth \pi\alpha}{1 + \coth^2 \pi \alpha \tan^2 \pi x})$$

The result now clearly follows. □

Lemma F: $\overline{\lim_{\alpha\to\infty}} \; \overline{\lim_{N\to\infty}} \| S_{X_{N,\alpha}} - B_A \|_1 = 0$.

Proof. For fixed $\alpha > 1$,

$$\| S_{X_{N,\alpha}} - B_A \|_1 \le 4 \overline{\lim_{N\to\infty}} \left| \max_{f \in [-1,1]} |S_{X_{N,\alpha}}| - \frac{A^2}{2} \right|$$

by Lemma A. By Lemma E,

$$\overline{\lim_{N\to\infty}} \left| \max_{f \in [-1,1]} |S_{X_{N,\alpha}}| - \frac{A^2}{2} \right| = \frac{A^2}{2} \left| \max_{x \in [0,1]} \frac{\sec^2 \pi x \coth \pi\alpha}{1 + \coth^2 \pi\alpha \tan^2 \pi x} \right|$$

Since $\sec^2 \pi x = 1 + \tan^2 \pi x$ and $\lim_{\alpha\to\infty} \coth \pi\alpha = 1$, we have

$$\overline{\lim_{\alpha\to\infty}} \left| \max_{x \in [0,1]} \frac{\sec^2 \pi x \coth \pi\alpha}{1 + \coth^2 \pi\alpha \tan^2 \pi x} - 1 \right| = 0$$

which completes the proof. □

REFERENCES

[1] A. Wyner, this book, Chapter III, Section 3.7.

[2] J.P. Kahane, "Sur les Polynomes a Coefficients Unimodulaire," *Bull. London Math Soc.*, 12, pp. 321-342 (1980).