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Comparison of peak and RMS gains for discrete-time systems

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Abstract: A convolution system can have a frequency response which is small for all frequencies, yet still greatly amplify the *peaks* of signals passing though it. For finite-dimensional systems, however, we establish the simple bound $\|h\|_1 \leq (2n+1)\|h\|_{\mathbf{H}^\infty}$, where $\|h\|_1$ is the peak gain of the system, $\|h\|_{\mathbf{H}^\infty}$ is the maximum frequency response of the system, and n its dimension. The same result for continuous-time systems is due to Gohberg and Doyle and it is mentioned in [6].

The bound implies that $\mathbf{H}^\infty$-optimal controllers, which minimize the maximum of some disturbance-to-error transfer function, cannot have very large peak gains from the disturbance to error.

Keywords: Peak gain, RMS gain, $\mathbf{H}_\infty$-norm, $\mathbf{l}_1$-norm, Hankel singular values.

1. Introduction

We consider the discrete-time convolution system $y = h * u$ where u (the *input*), y (the *output*) and h (the *impulse response*) are real-valued sequences on the nonnegative integers $\mathbb{Z}_+ \triangleq \{0, 1, 2\ldots\}$, and $h*u$ is defined by

$$y_k = \sum_{i=0}^{k} h_{k-i} u_i. \tag{1.0.1}$$

We now examine several different measures of the 'size' of a signal or the 'gain' of the convolution system (1.0.1).

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1.1. BIBO stability and peak gain

The peak or $\mathbf{l}^\infty$-norm of a signal u is defined by

$$\|u\|_\infty \triangleq \sup_{k\geq 0} |u_k|.$$

The set of bounded sequences, that is, those with finite peak, will be denoted $\mathbf{l}^\infty$, as usual. The system (1.0.1) will have the property that its output y is bounded whenever its input u is bounded if and only if

$$\sum_{i=0}^{\infty} |h_i| \triangleq \|h\|_1 < \infty \tag{1.1.1}$$

in which case we can bound the peak of the output by

$$\|y\|_\infty \leq \|h\|_1 \|u\|_\infty. \tag{1.1.2}$$

This last property (1.1.2) is called *bounded input bounded output* (BIBO) stability; a convolution system satisfying (1.1.1) is called BIBO stable. The bound (1.1.2) is in fact sharp, since there is a nonzero $u \in \mathbf{l}^\infty$ with

$$\|h * u\|_\infty = \|u\|_\infty \|h\|_1.$$

Thus $\|h\|_1$ is the *peak gain* of the convolution operator (1.0.1); it is the maximum factor by which the convolution operator can increase the peak of its input.

1.2. l^2 or RMS gain

Another useful norm on signals is the $\mathbf{l}^2$-norm

$$\|u\|_2 \triangleq \left(\sum_{i=0}^{\infty} u_i^2\right)^{1/2}$$

which may be interpreted as the square root of the total normalized energy in the signal u. $\mathbf{l}^2$ will denote the set of signals with finite $\mathbf{l}^2$-norm. The convolution operator (1.0.1) will map $\mathbf{l}^2$-signals

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into $\mathbf{l}^2$-signals if and only if the power series

$$H(\lambda) \triangleq \sum_{i=0}^{\infty} h_i \lambda^i \tag{1.2.1}$$

is analytic and bounded for (complex) $|\lambda| < 1$, that is,

$$\| h \|_{\mathbf{H}^\infty} \triangleq \sup_{|\lambda|<1} \left| \sum_{i=0}^{\infty} h_i \lambda^i \right| < \infty, \tag{1.2.2}$$

in which case the following bound holds:

$$\| y \|_2 \le \| h \|_{\mathbf{H}^\infty} \| u \|_2. \tag{1.2.3}$$

Such systems are called $\mathbf{l}^2$-stable.

The bound (1.2.3) is also sharp; there are $u \in \mathbf{l}^2$ with $\| y \|_2 / \| u \|_2$ as close as we want to $\| h \|_{\mathbf{H}^\infty}$ (but not equal, as in peak case, unless H has constant modulus on the unit circle). Hence $\| h \|_{\mathbf{H}^\infty}$ may be interpreted as the $\mathbf{l}^2$-gain of the convolution operator (1.0.1).

We remark here that the power series (1.2.1) need not converge for $|\lambda| = 1$, and that the requirement (1.2.2) is strictly weaker than (1), since $\| h \|_{\mathbf{H}^\infty} \le \| h \|_1$. Thus every BIBO convolution operator is $\mathbf{l}^2$-stable, but not vice versa. Examples of $\mathbf{l}^2$-stable but not BIBO stable convolution operators are quite contrived; one is given by

$$h_k \triangleq (k!)^{-1} H^{(k)}(0), \quad H(\lambda) = \mathrm{e}^{1/(\lambda-1)}. \tag{1.2.4}$$

When the convolution operator is also BIBO stable, the power series (1.2.1) does converge for $|\lambda| = 1$ and

$$\| h \|_{\mathbf{H}^\infty} = \sup_{\theta} | H(\mathrm{e}^{\mathrm{j}\theta}) |.$$

Thus the $\mathbf{l}^2$-gain $\| h \|_{\mathbf{H}^\infty}$ can be interpreted as the *maximum frequency response* of (1.0.1), that is, the maximum *steady state* response to sinusoidal inputs bounded by one. We note for later reference that the frequency response $|H(\mathrm{e}^{\mathrm{j}\theta})|$ of a BIBO stable convolution system is in fact a *continuous* function of θ.

Another interpretation of $\| h \|_{\mathbf{H}^\infty}$, perhaps more often appropriate, is as the RMS gain of (1.0.1). Let us define the RMS value of a signal u to be

$$\| u \|_{\mathrm{RMS}} \triangleq \limsup_{K\to\infty} \left(\frac{1}{K} \sum_{i=0}^{K-1} u_i^2 \right)^{1/2}.$$

Then we have $\| y \|_{\mathrm{RMS}} \le \| h \|_{\mathbf{H}^\infty} \| u \|_{\mathrm{RMS}}$, and this bound is sharp, indeed there is a nonzero u with $\| y \|_{\mathrm{RMS}} = \| h \|_{\mathbf{H}^\infty} \| u \|_{\mathrm{RMS}}$.

Note that $\| \cdot \|_{\mathrm{RMS}}$ is *not* a norm, but only a seminorm, since we can have nonzero signals with zero RMS value. For example, any 'transient' or decaying u ($u(k) \to 0$ as $k \to \infty$) has zero RMS value.

1.3. RMS response to white inputs

We say that a signal u is *white* if

$$\lim_{K\to\infty} \frac{1}{K} \sum_{i=0}^{K-1} u_i u_{i+m} = \begin{cases} 1 & m = 0, \\ 0 & m > 0. \end{cases}$$

Thus white signals have an RMS value of 1.

While $\| h \|_{\mathbf{H}^\infty}$ is the maximum RMS value of the output of (1.0.1) when the input signal has RMS value 1, if we restrict our attention to white input signals the RMS value of the output is always $\| h \|_2$, that is,

$$\| h * u \|_{\mathrm{RMS}} = \| h \|_2 \quad \text{for } u \text{ white.}$$

1.4. Relations among peak and RMS gains, RMS response to white inputs

We have seen three measures of the 'size' of the system (1.0.1): its peak and RMS gains, and its RMS response to white inputs. Intuition suggests that these measures are related. For example, it is tempting to conclude that if (1.0.1) has small RMS gain, it should have small peak gain, but this is simply not true.

In general nothing can be said other than

$$\| h \|_2 \le \| h \|_{\mathbf{H}^\infty} \le \| h \|_1.$$

Convolution systems with small RMS gain can have arbitrarily large peak gains, indeed infinite peak gain. The system (1.2.4) described in Section 1.2 has finite RMS gain

$$\| h \|_{\mathbf{H}^\infty} = \sup_{|\lambda|<1} | \mathrm{e}^{1/(\lambda-1)} | = \mathrm{e}^{-1/2}$$

but infinite peak gain, since as noted in Section 1.2, BIBO stable convolution systems have continuous frequency responses, yet for this example, $H(\mathrm{e}^{\mathrm{j}\theta})$ is not continuous at $\theta = 0$.

Similarly a convolution system can have small RMS response to white inputs but infinite RMS gain; an example is given by $H(\lambda) = (1-\lambda)^{-1/3}$, that is,

$$h_k \triangleq (k!)^{-1} H^{(k)}(0) = \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3k-2)}{3^{k+1}}.$$

This H is inbounded near $\lambda = 1$, hence $\| h \|_{\mathbf{H}^\infty} = \infty$, and so this convolution system can have infinite RMS response to an input with RMS value finite, say, 1. However, its RMS response to *white* inputs is finite, since

$$\begin{aligned}\| h \|_2^2 &= (2\pi)^{-1} \int_0^{2\pi} | H(e^{j\theta}) |^2 \, d\theta \\ &= \pi^{-1} 2^{-1/3} \int_0^{\pi} (1 - \cos\theta)^{-1/3} \, d\theta \\ &= \pi^{-1} 2^{-1/3} \int_0^{1} u^{-5/6} (2-u)^{-1/2} \, du \\ &\le \pi^{-1} 2^{2/3} 3.\end{aligned}$$

2. Controller and filter design

Many problems in controller and filter design can be cast in terms of making some 'error' convolution operator 'small' (see, e.g., [9]). Usually the error operator can be interpreted as mapping an input or disturbance to an error or output. Depending on our notion of size of a convolution operator, the goal of minimizing the error operator yields different controller or filter design schemes.

The linear quadratic Gaussian (LQG) controller minimizes the RMS response to white inputs of a certain (usually multi-input multi-output) error convolution operator. In [11], Zems pointed out that controller design schemes which minimize a *gain* ('multiplicative seminorm') have more desirable robustness properties than those which minimize a measure of the error operator which is not a gain, e.g. the RMS response to white inputs. He proposed to design controllers which minimize the RMS gain of the error operator, that is, the $\mathbf{H}^\infty$ norm of the error impulse response.

Implicit in such frequency-response methods is the assumption that in practice, an error operator small in the sense of maximum frequency response should be small in other sense, e.g. peak gain. The examples given in Section 1.4 show that this assumption does not hold generally, but we will show that a weak form of this assumption does hold in practice.

Recently Vidyasagar [10] proposed to design controllers which minimize the peak gain of an error operator, that is, the $\mathbf{l}^1$-norm of the error impulse response; Dahleh and Pearson [1] gave a solution to the $\mathbf{l}^1$-optimal controller design problem for discrete-time systems in 1985. A question which arises immediately is, how different can systems designed with LQG, $\mathbf{H}^\infty$- and $\mathbf{l}^1$-optimal controllers be? The examples of Section 1.4 suggest that they can be radically different.

3. Bounds for finite-dimensional systems

In most cases of practical interest, the impulse response h is that of a finite-dimensional dynamical system $\{A, b, c, d\}$:

$$\begin{aligned} x_{k+1} &= A x_k + b u_k, \\ y_k &= c x_k + d u_k, \end{aligned} \qquad x_0 = 0, \tag{3.1}$$

where $A \in \mathbb{R}^{n \times n}$, b, $c^{\mathrm{T}} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Thus $h_0 = d$, and for $k > 0$, $h_k = cA^{k-1}b$. For such impulse responses it is possible to bound the peak gain in terms of the RMS gain and n, the dimension of the state space.

Theorem 1. *If h comes from the dynamical system* (3.1), *then*

$$\| h \|_1 \le (2n+1) \| h \|_{\mathbf{H}^\infty}. \tag{3.2a}$$

If in addition $d = 0$, then

$$\| h \|_1 \le 2n \| h \|_{\mathbf{H}^\infty}. \tag{3.2b}$$

The continuous-time version of Theorem 1, which is identical, is a recent (unpublished) result of Gohberg and Doyle [6].

We will assume that the system is $\mathbf{l}^2$-stable, since otherwise the bounds above are vacuous. We will also assume that the system $\{A, B, c, d\}$ is minimal; this implies that the eigenvalues of A have magnitude less than one.

We mention that the bounds (3.2) are sharp, that is, for each n, there are dynamical systems with the ratios of peak to RMS gain as close as desired to $2n+1$. Allpass systems with widely spread dynamics have this property; for example consider

$$H(\lambda)=\prod_{k=1}^{n}\frac{\lambda-(1-\alpha^k)}{(1-\alpha^k)\lambda-1}$$

where $0<\alpha<1$. Of course $\|h\|_{\mathbf{H}^\infty}=1$, and as $\alpha\to 0$, $\|h\|_1\to 2n+1$. This last fact can be seen as follows: h consists of a sum of $2n+1$ decaying exponentials, alternating in sign, each of area one. As α becomes small, the exponentials have widely separated time scales, so that 'neighboring' exponentials do not cancel very much, thus $\|h\|_1$ is nearly $2n+1$.

We now turn to establishing Theorem 1. In fact the results (3.2) are implied by a sharper bound involving the *Hankel singular values* of the impulsive response h. Recall that the Hankel singular values $\sigma_{H1},\ldots,\sigma_{Hn}$ of (3.1) are the square roots of the eigenvalues of W_oW_c, where W_o and W_c are the observability and controllability Gramians of (3.1), i.e.

$$W_o\triangleq\sum_{k=0}^{\infty}A^{\mathrm{T}k}c^{\mathrm{T}}cA^k,\qquad W_c\triangleq\sum_{k=0}^{\infty}A^kbb^{\mathrm{T}}A^{\mathrm{T}k}$$

(these sums make sense because the eigenvalues of A have magnitude less than one). We order the Hankel singular values as usual:

$$\sigma_{H1}\geq\cdots\geq\sigma_{Hn}>0.$$

The σ_{Hi}'s depend only on the impulse response h and not on the particular realization $\{A, b, c, d\}$ of h, and so may be unambiguously called the Hankel singular values of h.

Theorem 2.

$$\|h\|_1\leq|d|+2(\sigma_{H1}+\cdots+\sigma_{Hn}).\qquad(3.3)$$

We postpone the proof of Theorem 2.

Lemma 1.

$$\|h\|_{\mathbf{H}^\infty}\geq\sigma_{H1}.$$

Lemma 1 is well known; it follows immediately from the characterizations

$$\|h\|_{\mathbf{H}^\infty}^2=\sup\left\{\sum_{i=0}^{\infty}y_i^2\,\middle|\,\sum_{i=0}^{\infty}u_i^2=1,\ y=h*u\right\},$$

$$\sigma_{H1}^2=\sup\left\{\sum_{i=k}^{\infty}y_i^2\,\middle|\,\sum_{i=0}^{\infty}u_i^2=1,\ u_k=u_{k+1}=\cdots=0,\ k\geq 0,\ y=h*u\right\}$$

(see e.g. [2] for the first and [5] for the second). Thus $\|h\|_{\mathbf{H}^\infty}^2\geq\sigma_{H1}^2$.

Theorem 1 follows immediately from Theorem 2 and Lemma 1:

$$\begin{aligned}\|h\|_1&\leq|d|+2\sum\sigma_{Hi}\\&\leq|d|+2n\sigma_{H1}\leq|d|+2n\|h\|_{\mathbf{H}^\infty}.\end{aligned}$$

Setting $d=0$ yields (3.2b). Noting that $\|h\|_{\mathbf{H}^\infty}\geq|H(0)|=|d|$ yields (3.2a).

We mention here that the sharper bound (3.3) shows that the order of n appearing in the bounds (3.2) can really be taken to be the *effective* order of the system, meaning the number of *significant* Hankel singular values, as opposed to the number of *nonzero* Hankel singular values.

Proof of Theorem 2. We have

$$\begin{aligned}\|h\|_1&=|d|+\sum_{k=0}^{\infty}|cA^kb|\\&=|d|+\sum_{k=0}^{\infty}|cA^{2k}b|+\sum_{k=0}^{\infty}|cA^{2k+1}b|.\end{aligned}\qquad(3.4)$$

Let us first consider the second term in (3.4). By the Cauchy–Schwarz inequality in $\mathbb{R}^n$,

$$|cA^{2k}b|\leq\|A^{\mathrm{T}k}c^{\mathrm{T}}\|_2\|A^kb\|_2,$$

so

$$\begin{aligned}\sum_{k=0}^{\infty}|cA^{2k}b|&\leq\sum_{k=0}^{\infty}\|A^{\mathrm{T}k}c^{\mathrm{T}}\|_2\|A^kb\|_2\\&\leq\left(\sum_{k=0}^{\infty}\|A^{\mathrm{T}k}c^{\mathrm{T}}\|_2^2\right)^{1/2}\left(\sum_{k=0}^{\infty}\|A^kb\|_2^2\right)^{1/2}\\&=(\operatorname{Tr}W_o\ \operatorname{Tr}W_c)^{1/2},\end{aligned}$$

using the Cauchy–Schwarz inequality in $\mathbf{l}^2$. Similarly the third term in (3.4) can be bounded above as

$$\sum_{k=0}^{\infty}|cA^{2k+1}b|\leq\sum_{k=0}^{\infty}\|A^{\mathrm{T}k}c^{\mathrm{T}}\|_2\|A^{k+1}b\|_2$$

$$\le \left(\sum_{k=0}^{\infty} \| A^{\mathrm{T}k} c^{\mathrm{T}} \|_2^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \| A^{k+1} b \|_2^2 \right)^{1/2}$$

$$= (\mathrm{Tr}\ W_o)^{1/2} \left(\mathrm{Tr} \sum_{k=1}^{\infty} A^k b b^{\mathrm{T}} A^{\mathrm{T}k} \right)^{1/2}$$

$$= (\mathrm{Tr}\ W_o)^{1/2} (\mathrm{Tr} W_c - \| b \|_2^2)^{1/2}$$

$$\le (\mathrm{Tr}\ W_o\ \mathrm{Tr}\ W_c)^{1/2}.$$

Thus from (3.4),

$$\| h \|_1 \le | d | + 2(\mathrm{Tr}\ W_o\ \mathrm{Tr}\ W_c)^{1/2}. \tag{3.5}$$

Now (3.5) is true for any realization $\{A, b, c, d\}$ of h; in particular for a balanced realization $\{A_b, b_b, c_b, d\}$ of h, we have

$$W_o = W_c = \mathrm{diag}[\sigma_{H1}, \dots, \sigma_{Hn}] \triangleq \Sigma_H.$$

Thus we have

$$\|h\|_1 \le |d| + 2(\sigma_{H1} + \cdots + \sigma_{Hn})$$

which establishes Theorem 2.

It is interesting to note that expressions similar to Tr W_o Tr W_c have appeared in recent work on sensitivity and overflow analysis of realizations of h [7,8].

It is worth mentioning that the balanced realization used above yields the best bound for $\| h \|_1$ based on the inequality (3.5). We will now show that among all realizations of h, Tr W_o Tr W_c has minimum value $\Sigma\sigma_{Hi}$, which is achieved by and only by any realization with W_o a multiple of W_c, the balanced realization a special case of this.

For any matrices F and G we have the inequality

$$\mathrm{Tr}(F^{\mathrm{T}} G) = \sum_{i,j} F_{ij} G_{ij} \le \left(\sum_{i,j} F_{ij}^2 \sum_{k,l} G_{kl}^2 \right)^{1/2}$$

$$= \left(\mathrm{Tr}(F^{\mathrm{T}} F)\ \mathrm{Tr}(G^{\mathrm{T}} G) \right)^{1/2}$$

with equality if and only if F and G are multiples of each other. Now if W_o and W_c are the Gramians of any (order n) realization of h, then they can be expressed as

$$W_o = T^{\mathrm{T}} \Sigma_H T, \qquad W_c = T^{-1} \Sigma_H T^{-\mathrm{T}}$$

where T is some nonsingular matrix (in fact, the coordinate transformation taking the balanced realization into the given realization). Applying the inequality above with $F = \Sigma_H^{1/2} T$ and $G = \Sigma_H^{1/2} T^{-\mathrm{T}}$ yields

$$(\mathrm{Tr}\ W_o\ \mathrm{Tr}\ W_c)^{1/2} \ge \mathrm{Tr}\left(T^{\mathrm{T}} \Sigma_H T^{-\mathrm{T}}\right)$$

$$= \mathrm{Tr}\ \Sigma_H = \sum \sigma_{Hi}. \tag{3.6}$$

Moreover equality obtains only when $\Sigma_H^{1/2} T$ is a multiple of $\Sigma_H^{1/2} T^{-\mathrm{T}}$, which is equivalent to $TT^{\mathrm{T}} = \alpha I$ for some constant α. This implies that $\alpha^2 W_o = W_c$; conversely if W_o is a multiple of W_c, then $TT^{\mathrm{T}} = \alpha I$ for some constant α, and equality obtains in (3.6). Thus the claim above is established.

Finally, we note that it is not possible to bound the RMS gain of a finite-dimensional system in terms of its RMS response to white inputs and its order. Let $0 < r < 1$, and consider the first-order system

$$x_{k+1} = r x_k + u_k, \quad y_k = x_k.$$

Thus $h_0 = 0$ and for $k > 0$, $h_k = r^{k-1}$, so

$$\| h \|_{\mathbf{H}^\infty} = (1 - r)^{-1} \quad \text{and} \quad \| h \|_2 = (1 - r^2)^{-1/2}.$$

For r near one, the ratio $\| h \|_{\mathbf{H}^\infty} / \| h \|_2$ is unbounded, establishing the impossibility of bounding this ratio in terms of the system order (here, one).

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