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PERTURBATION BOUNDS FOR STRUCTURED ROBUST STABILITY

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Abstract

"Robust stability" of a linear multivariable system, in the sense of robustness under multiplicative transfer function perturbations, is necessarily preserved under sufficiently small perturbations in the model parameters (i.e., it is a robust property). In this paper, parameter perturbation bounds ensuring the persistence of the robust stability property are derived.

I. INTRODUCTION

The frequency domain analysis of robust performance and stability of feedback systems with uncertainties is a subject which has attracted significant interest, prompted by the work of Doyle and Stein [1] . The early work resulted in necessary and sufficient conditions for robustness with respect to unstructured perturbations. These were expressed in terms of singular values of system transfer function matrices. Recently, Doyle [2] and Safonov [3] have initiated work on the case of structured uncertainties. Necessary and sufficient conditions for the satisfaction of robust performance and stability requirements in this setting are expressed in terms of the "structured singular value" [2] . Computational tools for checking these conditions for broad classes of systems are emerging.

Structured robust stability[1] involves large (but bounded) perturbations in certain loops of a multivariable system. That is, structured robust stability entails stability in the face of *large, structured perturbations*. In this paper, we study the persistence under *small, unstructured parametric perturbations* of the robust stability (structured or unstructured) of a system. The main goal is to derive parameter perturbation bounds which, when respected, ensure the robust stability of the perturbed models. The analysis of the paper employs results of [4] and [5],[6].

II. BACKGROUND

Consider a linear multivariable system subjected to structured uncertainty. That is, independent norm-bounded uncertainties, or perturbations, are assumed present in some, but not all, elements of the system. Denote by $H(s)$ the nominal transfer function matrix, which would completely characterize the system dynamics in the absence of the uncertainties. The transfer function matrix $H(s)$ is assumed to be a stable, proper, and rational representation of a causal linear system, all of whose entries have real coefficients. By rearranging the system, the hypothesized uncertainties may be grouped into a single block diagonal perturbation matrix [2]

$$\Delta(s) = \text{block diag}\,(\Delta_1(s),\ldots,\Delta_m(s)). \tag{1}$$

This transforms the original uncertain system representation S into the now well known standard block diagram representation S_Δ, depicted in Figure 1.

The notion of robust stability of system S with respect to structured uncertainty may be succinctly expressed in terms of the representation S_Δ. The following definition employs the norm

$$\|\Delta_i\| := \sup_{\omega\geq 0} \bar{\sigma}(\Delta_i(j\omega)), \tag{2}$$

[1] Doyle [2] has observed that a large class of robust *performance* questions may be restated as ones of robust stability. Therefore, we restrict our attention to the (structured) robust stability problem.

where $\bar{\sigma}(F)$ denotes the largest singular value of the matrix F.

Definition 1. By *structured robust stability* of the system S, one means stability of system S_Δ for any block diagonal $\Delta(s)$ as in (1) with $\|\Delta_i\| \leq 1$, $i = 1,\ldots,m$.

Following convention, the upper bound on $\|\Delta_i\|$, $i = 1,\ldots,m$, in this definition is taken to be 1; this simplifies notation while not resulting in a loss of generality. The uncertainty Δ is a *nonparametric uncertainty*, i.e. the class of all perturbations satisfying $\|\Delta_i\| \leq 1$, $i = 1,\ldots,m$ cannot be parametrized by a finite number of scalar parameters. In [4] , it is shown that structured robust stability relative to nonparametric uncertainty as in Definition 1 is equivalent to structured robust stability relative to certain *parametric uncertainties*, for the case of scalar blocks $\Delta_i(s)$. (The result may also be generalized to nonscalar blocks.) We recall the following theorem from Boyd [4] .

Theorem 1 [4]. *Let the blocks $\Delta_i(s)$ be scalars. Then system S possesses the structured robust stability property of Definition 1 if and only if S_Δ is stable for all $\Delta(s) = \text{diag}\,(\Delta_1(s),\ldots,\Delta_m(s))$, where each $\Delta_i(s)$ is of the form*

$$\Delta_i(s) = \pm\frac{s-d_i}{s+d_i} \text{ or } \pm 1 \tag{3}$$

and each $d_i > 0$, $i = 1,\ldots,m$.

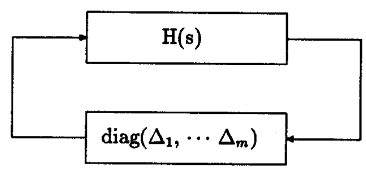

Figure 1

Theorem 1 allows us to study a parametric stability problem, parametrized by d_i, $i = 1,\ldots,m$, in place of the original robust stability problem, which involved nonparametric uncertainties. Note that the third and fourth cases in Theorem 1 result from taking the limit as $d_i \rightarrow 0$ in the first and second cases, respectively.

The results of Abed [5] deal with parametric stability problems, and were motivated by problems in multiparameter singular perturbation theory. The notion of D-stability of a matrix, introduced by Enthoven and Arrow [6] in the context of competitive market equilibrium, arises naturally in this context as well. Recall [7] that a matrix $F \in R^{m\times m}$ is said to be *D-stable* if DF is stable for each diagonal matrix D with positive diagonal entries. Khalil and Kokotovic [8] generalized this concept to one of "block D-stability," wherein D is restricted to be of the form $(d_1 I_{m_1}, \ldots, d_M I_{m_M})$. In this case, we say F is block D-stable with respect to the index $(m_1, \ldots, m_M)$. Abed [5] showed by example that D-stability need not be robust to arbitrarily small perturbations of a matrix, and introduced the following related concept.

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Definition 2. The matrix $F \in R^{m \times m}$ is *strongly D-stable* if there is a $\mu > 0$ such that $F+G$ is D-stable for each $G \in R^{m \times m}$ with $\|G\| < \mu$.

The following notion of strong block D-stability generalizes the strong D-stability idea in the same way that block D-stability relates to D-stability.

Definition 3. The matrix $F \in R^{m \times m}$ is *strongly block D-stable* (with respect to the multi-index $(m_1, \ldots, m_M)$) if there is a $\mu > 0$ such that $F+G$ is block D-stable (with respect to the same multi-index) for each $G \in R^{m \times m}$ with $\|G\| < \mu$.

A theorem from [5] is now recalled, to illustrate the implications of strong D-stability in multiparameter singular perturbation problems. Although this result is not used explicitly in this paper, its relationship with the current problem will become clear in the next section. Consider a linear time-invariant system

$$\dot{x} = A_{11}x + A_{12}y \tag{4a}$$

$$E(\epsilon)\dot{y} = A_{21}x + A_{22}y, \tag{4b}$$

where $E(\epsilon) := \text{diag}(\epsilon_1, \ldots, \epsilon_m)$. Here $x \in R^n$, $y \in R^m$, and the A_{ij} are real matrices of conformable dimension. The *singular perturbation parameters* ϵ_i, $i = 1,\ldots,m$ are small and positive.

Theorem 2 [5] . *Suppose that all eigenvalues of* $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$ *have strictly negative real parts, and let* A_{22} *be strongly D-stable. Then there is a* $\mu > 0$ *such that the null solution of system* (4) *is asymptotically stable for all* $\epsilon_i > 0$, $i = 1,\ldots,m$, $\|\epsilon\| < \mu$.

III. EXPLICIT FINITE DIMENSIONAL REALIZATION

Consider the system S with transfer function matrix $H(s)$ strictly proper. Let $H(s)$ have a realization

$$\dot{x} = Ax + Bu \tag{5a}$$

$$y = Cx \tag{5b}$$

containing no unstable hidden modes. To derive general parameter perturbation bounds for robust stability, we first explicate Theorem 1 as it applies to (5).

The following notation will be needed. Denote by α the ordered set of integers i for which $\Delta_i(s) = \pm 1$, and write

$$\alpha = \{k_1, k_2, \ldots, k_a\} \tag{6}$$

where $a \leq m$ and $k_1 \leq k_2 \leq \cdots \leq k_a$. Similarly, denote by β the ordered set of integers i for which $\Delta_i(s) = \pm \frac{s - d_i}{s + d_i}$. Write

$$\beta = \{l_1, l_2, \ldots, l_b\} \tag{7}$$

where $b = m - a$ and $l_1 \leq l_2 \leq \cdots \leq l_b$. Denote by C_i the i-th *row* of C, and by B_i the i-th *column* of B. Let u_α (respectively y_α) be the vector of components of u (respectively y) whose index belongs to α, and analogously for the notation u_β, y_β. Denote by B_α the matrix of columns of B (respectively, rows of C) whose index belongs to α, arranged according to increasing order of the index. Similarly, introduce the notation B_β, C_β, for the remaining columns and rows, respectively, of B and C, respectively. Finally, use J_α to denote any diagonal $a \times a$ matrix all of whose diagonal entries are 1 or -1; similarly, J_β denotes any diagonal $b \times b$ matrix all of whose diagonal entries are 1 or -1.

For $i \in \alpha$, $u_i = \pm y_i = \pm C_i x$. Hence, using the notation introduced above, we have

$$u_\alpha = J_\alpha C_\alpha x. \tag{8}$$

For $i \in \beta$, $\Delta_i(s) = \pm \frac{s - d_i}{s + d_i}$. This implies, for $1 \leq i \leq b$,

$$\dot{u}_{l_i} + d_i u_{l_i} = \pm(\dot{y}_{l_i} - d_i y_{l_i}). \tag{9}$$

Define

$$z_i := \frac{y_{l_i} \mp u_{l_i}}{2}, \tag{10}$$

where $\mp$ means that the - sign is used if the + sign prevails in (9), and the + sign is used otherwise. This yields

$$\dot{z}_i = d_i(y_{l_i} - z_i)$$

$$= d_i(C_{l_i}x - z_i), \tag{11}$$

regardless of the sign appearing on the right side of (9). Define $D_\beta = \text{diag}(d_1, \ldots, d_b)$, which is simply an arbitrary positive diagonal matrix, and denote by z the column vector $z := (z_1, \ldots, z_b)^T$. Then (11) implies z is governed by the dynamics

$$\dot{z} = D_\beta(C_\beta x - z). \tag{12}$$

Eq. (10) also implies that for $1 \leq i \leq b$,

$$u_{l_i} = \pm y_{l_i} \mp 2z_i. \tag{13}$$

Hence, the vector u_β is given by

$$u_\beta = J_\beta(y_\beta - 2z)$$

$$= J_\beta(C_\beta x - 2z). \tag{14}$$

Now, x is governed by the dynamics

$$\dot{x} = Ax + Bu$$

$$= Ax + B_\alpha u_\alpha + B_\beta u_\beta$$

$$= Ax + B_\alpha J_\alpha C_\alpha x + B_\beta J_\beta (C_\beta x - 2z)$$

$$= (A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta)x - 2B_\beta J_\beta z \tag{15}$$

Summarizing, the overall system dynamics is given by

$$\dot{x} = (A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta)x - 2B_\beta J_\beta z \tag{16a}$$

$$\dot{z} = D_\beta(C_\beta x - z). \tag{16b}$$

Eq. (16) may be written more compactly as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta & -2B_\beta J_\beta \\ D_\beta C_\beta & -D_\beta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}. \tag{17a}$$

The coefficient matrix in Eq. (17a) can be factored as

$$\begin{pmatrix} I & 0 \\ 0 & D_\beta \end{pmatrix} \begin{pmatrix} A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta & -2B_\beta J_\beta \\ C_\beta & -I \end{pmatrix} \tag{17b}$$

We are now in a position to give a more explicit restatement of Theorem 1 in terms of block D-stability. We then use this statement to derive parameter perturbation bounds for the persistence of structured robust stability. The following result is obtained easily from the preceding analysis using the notion of block D-stability. Here n is the dimension of x.

Corollary 1. *System S is robustly stable with respect to the structured uncertainty* $\Delta(s) = \text{diag}(\Delta_1(s),\ldots,\Delta_m(s))$ *if and only if the following matrix is block D-stable with respect to the index* $(n,1,1,\ldots,1)$, *for all ordered partitions* α, β *of the set* $\{1,\ldots,m\} = \alpha \cup \beta$, $\alpha \cap \beta = \emptyset$, *and for all choices of the diagonal matrices* J_α, J_β *with only* 1 *and* -1 *occurring in the diagonals:*

$$Z_{\alpha\beta} := \begin{pmatrix} A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta & -2B_\beta J_\beta \\ C_\beta & -I \end{pmatrix}. \tag{18}$$

Since the structured robust stability property addressed in Corollary 1 persists under all sufficiently small perturbations in the system matrices A, B, C, we have the following additional result, interesting in its own right.

Corollary 2. *If the matrix $Z_{\alpha\beta}$ is block D-stable with respect to the index $(n,1,1,\ldots,1)$, for all ordered partitions α, β of the set $\{1,\ldots,m\} = \alpha\cup\beta$, $\alpha\cap\beta = \emptyset$, and for all choices of the diagonal matrices J_α, J_β with only 1 and -1 occurring in the diagonals, then this property is preserved under sufficiently small perturbations of the system matrices A, B, C.*

IV. PARAMETER PERTURBATION BOUNDS

By Corollary 2, the matrix $Z_{\alpha\beta}$ of Eq. (18) is block D-stable and remains so under small perturbations of the matrices A, B, C for each α if and only if the original system is robustly stable. This suggests that $Z_{\alpha\beta}$ may indeed be strongly block D-stable. It was shown by Abed [5] that a matrix F is strongly block D-stable if there exists a symmetric positive definite block diagonal matrix $P = (P_1, \ldots, P_M)$ such that Q given by

$$PF + F^T P = -Q \tag{19}$$

is positive definite. Moreover, an upper bound on the norm of the perturbation G to F ensuring retention of block D-stability is given by

$$\|G\| < \frac{\|Q\|}{2\|P\|}. \tag{20}$$

Here the norm is any symmetric matrix norm. Applying this to the matrix $Z_{\alpha\beta}$ for each α and J_β, and taking the minimum over α and J, we arrive at an upper bound for robust stability. The details of this are worked out next, and a computation using the Kalman-Yacubovitch Lemma is introduced.

If, for each partition (α,β), there is a positive definite matrix $P_{11,\alpha\beta}$ and a positive *diagonal* matrix $P_{22,\alpha\beta}$ such that $Q_{\alpha\beta}$ given by

$$\begin{pmatrix} P_{11,\alpha\beta} & 0 \\ 0 & P_{22,\alpha\beta} \end{pmatrix} Z_{\alpha\beta} + Z_{\alpha\beta}^T \begin{pmatrix} P_{11,\alpha\beta} & 0 \\ 0 & P_{22,\alpha\beta} \end{pmatrix} = -Q_{\alpha\beta} \tag{21}$$

is positive definite, then Eq. (19) is satisfied for each α,β, J_α, J_β, and Eq. (20) gives an upper bound on parameter perturbations for robust stability. Namely, if each perturbation matrix $G_{\alpha\beta}$, now understood as a perturbation of the matrix $Z_{\alpha\beta}$, is bounded in norm as follows:

$$\|G_{\alpha\beta}\| < \frac{\|Q_{\alpha\beta}\|}{2(\|P_{11,\alpha\beta}\| + \|P_{22,\alpha\beta}\|)}, \tag{22}$$

then robust stability of the perturbed system is ensured. Moreover, it is straightforward to state upper bounds on the perturbations of the system matrices A, B, C which guarantee the inequality (22).

Eq. (21) may be interpreted in terms of positive real functions using the Kalman-Yacubovitch Lemma. To see this, rewrite (21) as

$$Q_{\alpha\beta} = \begin{pmatrix} -P_{11,\alpha\beta}\overline{A} - \overline{A}^T P_{11,\alpha\beta} & -P_{11,\alpha\beta}\overline{B} - \overline{C}^T P_{22,\alpha\beta} \\ -\overline{B}^T P_{11,\alpha\beta} - P_{22,\alpha\beta}\overline{C} & 2P_{22,\alpha\beta} \end{pmatrix} \tag{23}$$

where $\overline{A} := A + B_\alpha J_\alpha C_\alpha + B_\beta J_\beta C_\beta$, $\overline{B} := -2B_\beta J_\beta$, and $\overline{C} := C_\beta$.

In the case $\alpha = \{1,\ldots,m\}$, the existence of a positive definite matrix $P_{11,\alpha\beta}$ ensuring positive definiteness of $Q_{\alpha\beta}$ reduces to a study of the Liapunov matrix equation $P_{11,\alpha\beta}\overline{A} + \overline{A}^T P_{11,\alpha\beta} = -Q_{\alpha\beta}$. The analysis of this equation, including derivation of associated perturbation bounds, is standard. Now assume that for each *proper* subset α of $\{1,\ldots,m\}$ and matrix J_α, the pair $(A + B_\alpha J_\alpha C_\alpha, B_\beta)$ is controllable, and the pair $(A + B_\alpha J_\alpha C_\alpha, C_\beta)$ is observable. (Recall that β is determined once α has been specified.) Then using the PBH rank tests, it is straightforward to show that, for each such α, the pairs $(\overline{A},\overline{B})$, $(\overline{A},\overline{C})$ are controllable and observable, respectively. This implies that for each such α, the representation $(\overline{A},\overline{B},-P_{22,\alpha\beta}\overline{C},P_{22,\alpha\beta})$ is also minimal, for any positive diagonal $P_{22,\alpha\beta}$. From the Kalman-Yacubovitch Lemma, it follows that if there exists a positive diagonal matrix $P_{22,\alpha\beta}$ such that the transfer function

$$R(s) := P_{22,\alpha\beta}\{I - \overline{C}(sI-\overline{A})^{-1}\overline{B}\} \tag{24}$$

is strictly positive real, then there is a positive definite $P_{11,\alpha\beta}$ such that $Q_{\alpha\beta}$ given by (23) is positive definite. The multivariable Kalman-Yacubovitch Lemma in the form presented by Anderson, or more simply that given by Boyd [9] , may be used to obtain this result.

V. CONCLUSIONS

We have shown how parameter perturbation bounds for the classical frequency domain notion of robust stability with respect to multiplicative perturbations may be computed. The derivation of the bounds was performed in the time domain, using results on strongly block D-stable matrices. The assumption needed to apply the computation was related to the positive realness of a related multivariable transfer function.

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