Control of Uncertainty-Affected Discrete Time Linear Systems via Convex Programming

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Abstract

In [1], we have demonstrated that robust optimization of linear finite-horizon control in a discrete time linear dynamical system affected by uncertain disturbances becomes computationally tractable if, rather than using natural parameterization of a linear output-based control law, one passes to a specific re-parameterization of the law, representing it as affine control law based on the so called "purified outputs". With the traditional parameterization, the states and the controls, being affine in the initial state and the disturbances, are highly nonlinear in the parameters of the control law; with the new parameterization, the states and the controls become *bi-affine*, that is, affine in the initial state and the disturbances, the parameters of the control law being fixed, and affine in the parameters of the control law, the disturbances and the initial state being fixed. As a result, synthesis of a finite-horizon control law satisfying, in a robust fashion, a given system of linear constraints on the finite-horizon state-control trajectory, reduces to solving an explicit convex program and thus becomes computationally tractable. In this follow up paper we extend the above methodology to optimizing infinite-horizon control in a time-invariant linear system and illustrate our methodology on examples involving discrete time H_∞- and L_1-control.

1 Introduction

Consider a discrete time linear dynamical system

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& A_t x_t + B_t u_t + R_t d_t,\ t = 0, 1, ... \\ y_t &=& C_t x_t + D_t d_t \end{array} \tag{1.1}$$

where x_t are states, y_t are outputs, u_t are controls, and d_t are external disturbances at time t. In [1], we have associated with system (1.1) "closed" by an arbitrary control law $u_t = U_t(y_0, y_1, ..., y_t)$

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the sequence of *purified outputs* v_t given by the recurrence

$$\begin{array}{rcl} \widehat{x}_0 & = & 0 \\ \widehat{x}_{t+1} & = & A_t\widehat{x}_t + B_t u_t \\ \widehat{y}_t & = & C_t\widehat{x}_t \\ v_t & = & y_t - \widehat{y}_t \end{array} \tag{1.2}$$

and have shown that the purified outputs v_t are in fact independent of the control law and are affine functions of $(z, d^t = (d_0, ..., d_t))$ with coefficients readily given by $\{A_\tau, B_\tau, C_\tau, D_\tau, R_\tau\}_{\tau=0}^t$. Furthermore, since v_t is observable at time t when the decision on u_t should be made, we can use *purified-output-based affine control laws* - laws of the form

$$u_t = h_t + \sum_{\tau=0}^{t} H_{t\tau} v_\tau. \tag{1.3}$$

We have proved in [1] that as far as state-control behaviour of the closed loop system is concerned, control laws (1.3) are exactly equivalent to traditional output-based affine control laws

$$u_t = g_t + \sum_{\tau=0}^{t} G_{t\tau} y_\tau. \tag{1.4}$$

Specifically, with both types of laws, the states x_t and the controls u_{t-1} in the closed-loop system become affine functions of (z, d^{t-1}); the collection of these functions given by a control law (1.3) can be obtained from a control law in the form of (1.4), and vice versa[1)]. However, representation (1.3) of an affine control law has a significant advantage as compared with representation (1.4): with the former representation, states and controls in the closed loop system *are affine* in the parameters $\{h_t, \{H_{t\tau}\}$ of law (1.3), whereas with the latter representations, states and controls are highly nonlinear in the parameters $\{g_t, G_{t\tau}\}$ of law (1.4).

The fact that the purified-output-based control laws (1.3) generate state-control trajectories which are *bi-affine* in $\{z, d_t\}_{t\geq 0}$ and in $\{h_t, H_{t\tau}\}_{t\geq\tau\geq 0}$ (that is, x_t, u_{t-1} are affine in $\{z, d_t\}$ when $\{h_t, H_{t\tau}\}$ is fixed and are affine in $\{h_t, H_{t\tau}\}$ when $\{z, d_t\}$ is fixed) is of major significance. Indeed, consider the problem of synthesizing a finite-horizon affine control law ensuring that the resulting finite horizon state-control trajectory satisfies, in a robust w.r.t. $\{z, d_t\}$ fashion, a system of linear constraints. Due to bi-affinity, this problem reduces to an explicit convex program (sometimes, just a linear one) even with a pretty general interpretation of what “robust w.r.t. $\{z, d_t\}$ fashion” means (for details, see [1]).

In this follow-up paper, we focus on time-invariant dynamical systems (1.1) and “(nearly) time-invariant” purified-output-based affine control laws and demonstrate that some *infinite-horizon* control problems, such as discrete time H_∞/L_1-synthesis, associated with these laws admit explicit convex reformulation and thus can be solved efficiently.

[1)] For the particular case $y_t \equiv x_t$, a similar equivalence result was independently established in the recent paper [3].

2 Time invariant purified-output-based affine control laws

From now on we focus on a time-invariant version of (1.1)

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& Ax_t + Bu_t + Rd_t,\ t = 0, 1, ... \\ y_t &=& Cx_t + Dd_t \end{array} \tag{2.5}$$

We make the following crucial assumption:

> **Assumption A** *The open-loop system* (2.5) *is stable, that is, the spectral radius of A is less than 1.*

Note that what follows can be straightforwardly extended to the case when (2.5) is unstable, but we are given a stabilizing time-invariant linear feedback for (2.5) – a matrix K such that the spectral radius of $A + BKC$ is < 1 (see Section 5).

Let us call a control law (1.3) *time-invariant of order k*, $k = 1, 2, ...$, if

$$u_t = \sum_{s=0}^{k-1} H_s v_{t-s}; \tag{2.6}$$

here H_0,...,H_{k-1} are fixed matrices of appropriate sizes and by definition $v_\tau = 0$ for $\tau < 0$ (the latter convention is in force for all subsequent quantities with negative indices). Let x_t, u_t, y_t, $\widehat{x}_t$, $\widehat{y}_t$ and v_t be the entities associated with the control law (2.6) according to (2.5) and (1.2), and let $\xi_t = x_t - \widehat{x}_t$. Then

$$\begin{array}{rcl} \xi_0 &=& z \\ \xi_{t+1} &=& A\xi_t + Rd_t \\ v_t &=& C\xi_t + Dd_t. \end{array}$$

Let us set[2)]

$$\begin{array}{l} \xi_t^k = [\xi_t; \xi_{t-1}; ...; \xi_{t-k+1}],\ v_t^k = [v_t; v_{t-1}; ...; v_{t-k+1}],\ d_t^k = [d_t; d_{t-1}; ...; d_{t-k+1}], \\ A^{(k)} = \mathrm{Diag}\{\underbrace{A, ..., A}_{k}\},\ R^{(k)} = \mathrm{Diag}\{\underbrace{R, ..., R}_{k}\},\ C^{(k)} = \mathrm{Diag}\{\underbrace{C, ..., C}_{k}\}, D^{(k)} = \mathrm{Diag}\{\underbrace{D, ..., D}_{k}\}, \\ H = [H_0, H_1, ..., H_{k-1}],\ w_t^k = [x_t; \xi_t^k]. \end{array}$$

It is immediately seen that the evolution of ξ_t^k, v_t^k, u_t with t resulting from the control law (2.6) is given by

$$\begin{array}{rcl} \xi_{t+1}^k &=& A^{(k)}\xi_t^k + R^{(k)}d_t^k \\ v_t^k &=& C^{(k)}\xi_t^k + D^{(k)}d_t^k \\ u_t &=& Hv_t^k \end{array} \tag{2.7}$$

whence the evolution of w_t^k is given by

$$w_{t+1}^k \equiv \left[\begin{array}{c} x_{t+1} \\ \xi_{t+1}^k \end{array}\right] = \underbrace{\left[\begin{array}{c|c} A & BHC^{(k)} \\ \hline & A^{(k)} \end{array}\right]}_{A_+} \underbrace{\left[\begin{array}{c} x_t \\ \xi_t^k \end{array}\right]}_{w_t^k} + \underbrace{\left[\begin{array}{c|c} R & BHD^{(k)} \\ \hline & R^{(k)} \end{array}\right]}_{R_+} \underbrace{\left[\begin{array}{c} d_t \\ d_t^k \end{array}\right]}_{d_t^+} \tag{2.8}$$

[2)] We use "MATLAB notation": for matrices $B_1, ..., B_m$ with the same number of columns, $[B_1; B_2; ...; B_m]$ is the matrix obtained by writing B_2 under B_1, B_3 under B_2, etc., while for matrices $B_1, ..., B_m$ with the same number of rows, $[B_1, ..., B_m]$ is the matrix obtained by writing B_2 to the right of B_1, B_3 to the right of B_2, etc.

Since $A^{(k)}$ is block-diagonal, with diagonal blocks equal to A, we arrive at the following observation:

Proposition 1 *System* (2.8) *is stable independently of the choice of* H, *and the resolvent* $\mathcal{R}_{A_+}(z) = (zI - A_+)^{-1}$ *of* A_+ *is affine in* H:

$$\mathcal{R}_{A_+}(z) = \left[\begin{array}{c|c|c|c|c} \mathcal{R}_A(z) & \mathcal{R}_A(z)BH_0C\mathcal{R}_A(z) & \mathcal{R}_A(z)BH_1C\mathcal{R}_A(z) & \dots & \mathcal{R}_A(z)BH_{k-1}C\mathcal{R}_A(z) \\ \hline & \mathcal{R}_A(z) & & & \\ \hline & & \mathcal{R}_A(z) & & \\ \hline & & & \ddots & \\ \hline & & & & \mathcal{R}_A(z) \end{array}\right] \tag{2.9}$$

where $\mathcal{R}_A(z) = (zI - A)^{-1}$.

It follows that *if we are given design specifications which are convex in* $\mathcal{R}_{A_+}(\cdot)$, *we can easily find* H *such that* $\mathcal{R}_{A_+}(\cdot)$ *complies with these specifications, or easily find out that no such* H *exists.*

3 Example: discrete time H_∞ synthesis

When the disturbances d_t in (2.8) form a "geometric progression" $d_t = z^t d$, $z \in \mathbf{C}$, with z different from the eigenvalues of A and from 0, then every solution to the stable system (2.8) approaches, as $t \to \infty$, the "steady-state" solution

$$w_t^k = z^t W(z) d,$$

where

$$W(z) = \mathcal{R}_{A_+}(z) \left[\begin{array}{c|c} R & BHD^{(k)} \\ \hline & R^{(k)} \end{array}\right] \begin{bmatrix} I \\ I \\ z^{-1}I \\ z^{-2}I \\ \vdots \\ z^{1-k}I \end{bmatrix} \tag{3.10}$$

In particular, the "steady-state" behaviour of states x_t and controls u_t is given by

$$\begin{array}{rcl} x_t & = & z^t \mathcal{H}_x(z) d, \\ & & \mathcal{H}_x(z) = \mathcal{R}_A(z) \left[R + \sum\limits_{s=0}^{k-1} z^{-s} BH_s \left[D + C\mathcal{R}_A(z)R\right]\right]; \\ u_t & = & z^t \mathcal{H}_u(z) d, \\ & & \mathcal{H}_u(z) = \left[H_0 + z^{-1}H_1 + z^{-2}H_2 + \dots + z^{1-k}H_{k-1}\right]\left[D + C\mathcal{R}_A(z)R\right]. \end{array} \tag{3.11}$$

Both "transfer functions" $\mathcal{H}_x(z)$, $\mathcal{H}_u(z)$ are of the form $p^{-1}(z)P(z,H)$, where $p(z)$ is a scalar polynomial independent of H, and $P(z,H)$ is a matrix-valued polynomial of z with coefficients *affinely depending on* H. It follows that *the problem of choosing* H *in a way which ensures that* $\mathcal{H}_x(\cdot)$, $\mathcal{H}_u(\cdot)$ *satisfy a system of convex constraints is a convex program.* For example, assume that design specifications require from the "full" transfer function

$$\mathcal{H}_{xu}(z) = [\mathcal{H}_x(z); \mathcal{H}_u(z)] \tag{3.12}$$

to satisfy a finite system of constraints of the form

$$\|Q_i(z) - M_i(z)\mathcal{H}_{xu}(z)N_i(z)\| \le \tau_i \quad \forall (z = \exp\{\imath\omega\} : \omega \in \Delta_i), \tag{3.13}$$

where $Q_i(z)$, $M_i(z)$, $N_i(z)$ are given rational matrix-valued functions with no singularities on the unit circumference, $\Delta_i \subset [0, 2\pi]$ are given segments, and $\|\cdot\|$ is the standard matrix norm (the largest singular value). It is easy to see that each of the constraints (3.13) is semidefinite-representable, that is, can be expressed equivalently by an explicit system of LMI's in variables H, τ and appropriate additional variables; as a result, the problem in question can be posed as an explicit semidefinite program.

To verify that constraints (3.13) are semidefinite-representable, note that these constraints are of the generic form

$$\|p^{-1}(z)P(z, H)\| \le \tau \,\forall (z = \exp\{\imath\omega\} : \omega \in \Delta), \tag{3.14}$$

where $p(\cdot)$ is a scalar polynomial independent of H and $P(z, H)$ is a polynomial in z with $m \times n$ matrix coefficients affinely depending on H. Constraint (3.14) can be expressed equivalently by the semi-infinite matrix inequality

$$\begin{bmatrix} \tau I_m & P(z, H)/p(z) \\ (P(z, H))^*/(p(z))^* & \tau I_n \end{bmatrix} \succeq 0 \quad \forall (z = \exp\{\imath\omega\} : \omega \in \Delta)$$

(* stands for the Hermitian conjugate, $\Delta \subset [0, 2\pi]$ is a segment) or, which is the same,

$$S_{H,\tau}(\omega) \equiv \begin{bmatrix} \tau p(\exp\{\imath\omega\})(p(\exp\{\imath\omega\}))^* I_m & (p(\exp\{\imath\omega\}))^* P(\exp\{\imath\omega\}, H) \\ p(\exp\{\imath\omega\})(P(\exp\{\imath\omega\}, H))^* & \tau p(\exp\{\imath\omega\})(p(\exp\{\imath\omega\}))^* I_n \end{bmatrix} \succeq 0 \quad \forall \omega \in \Delta.$$

Observe that $S_{H,\tau}(\omega)$ is a trigonometric polynomial taking values in the space of Hermitian matrices of appropriate size, the coefficients of the polynomial being affine in H, τ. It is known (see [2]) that the cone $\mathcal{P}_m$ of (coefficients of) all Hermitian matrix-valued trigonometric polynomials $S(\omega)$ of degree $\le m$ which are $\succeq 0$ for all $\omega \in \Delta$ is semidefinite representable: there exists an explicit LMI

$$\mathcal{A}(S, u) \succeq 0$$

in variables S (the coefficients of a polynomial $S(\cdot)$) and additional variables u such that $S(\cdot) \in \mathcal{P}_m$ if and only if S can be extended by appropriate u to a solution of the LMI. Consequently, the relation

$$\mathcal{A}(S_{H,\tau}, u) \succeq 0, \tag{$*$}$$

which is an LMI in H, τ, u, is a semidefinite representation of (3.13): H, τ solve (3.13) if and only if there exists u such that H, τ, u solve $(*)$.

3.1 Comparison with linear feedback

We have demonstrated that rather general control problems related to robust behaviour of time invariant linear dynamical systems can be easily solved via convex optimization, provided that the candidate control laws are time invariant purified-output-based ones and that the open loop system is stable (or can be made so by applying a given time invariant linear output-based feedback). An immediate question arises: what is the "power" of our family of control laws? We intend to demonstrate that in a sense it is *at least* as strong as the one of the usual output-based time invariant linear feedback controls. Indeed, consider a *stable* open-loop dynamical system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + Rd_t \\ y_t &= Cx_t + Dd_t \end{aligned} \tag{3.15}$$

"closed" by a feedback

$$u_t = Ky_t, \tag{3.16}$$

and assume that the closed loop system is stable as well. In this case, with $d_t = \zeta^t d$ and ζ different from the eigenvalues of A and from 0, every solution $\{x_t\}_{t=0}^{\infty}$ to (3.15) - (3.16) approaches, as $t \to \infty$, the steady-state solution $\widetilde{x}_t = \zeta^t \mathcal{R}_{A_K}(\zeta)[BKD + R]d$, where $\mathcal{R}_{A_K}(\zeta) = (\zeta I - [A + BKC])^{-1}$ is the resolvent of the matrix $A_K = A + BKC$. Consequently, as $t \to \infty$, the sequence of associated controls u_t approaches the steady-state control sequence

$$\widetilde{u}_t = \zeta^t K \left[D + C\mathcal{R}_{A_K}(\zeta)[BKD + R]\right] d.$$

In other words, the full transfer function of (3.15) – (3.16) is

$$\widetilde{\mathcal{H}}_{xu}(\zeta) = \left[\widetilde{H}_x(\zeta); \widetilde{H}_u(\zeta)\right] = \left[\mathcal{R}_{A_K}(\zeta)[BKD + R]; K\left[D + C\mathcal{R}_{A_K}(\zeta)[BKD + R]\right]\right]. \tag{3.17}$$

The announced "strength" of time invariant purified-output-based affine control laws as compared to linear feedback ones can roughly be expressed by the following claim: *outside of the set comprised of 0, the eigenvalues of A and the eigenvalues of $A + BKC$, the full transfer function* (3.17) *of the feedback-based closed loop system in question can be approximated to any desired accuracy, uniformly on compact sets, by transfer functions yielded by time invariant purified-output-based affine control laws.* The precise statement is as follows.

Proposition 2 *Let A and $A+BKC$ be discrete time stable matrices, and let γ be a closed Gordan curve belonging to the interior D_o of the unit disk in the complex plane $\mathbf{C}$ and such that the "inner part" $\Gamma_{\rm i}$ of γ (the bounded open set in $\mathbf{C}$ with the boundary γ) contains the set Λ comprised of 0 and all eigenvalues of A and $A + BKC$ (since $A + BKC$ and A are discrete time stable, we have $\Lambda \subset D_o$). Let, further, $\Gamma_{\rm o}$ be the complement of* int $\Gamma_{\rm i}$ *in the complex sphere $\mathbf{C}_* = \mathbf{C} \cup \{\infty\}$. Then for every $\epsilon > 0$ there exists $k = k(\epsilon)$ and collection $H^k = [H_0, ..., H_{k-1}]$ of matrices such that the transfer function $\mathcal{H}_{xu}(\cdot)$ associated, by virtue of* (3.11) and (3.12), *with the time invariant purified-output-based control law* (2.6) *given by H^k satisfies the relation*

$$\|\widetilde{\mathcal{H}}_{xu}(\zeta) - \mathcal{H}_{xu}(\zeta)\| \leq \epsilon \;\; \forall \zeta \in \Gamma_{\rm o}.$$

Proof. Both the resolvents $\mathcal{R}_A(\zeta) = (\zeta I - A)^{-1}$ and $\mathcal{R}_{A_K}(\zeta) = (\zeta I - A - BKC)^{-1}$ are rational matrix-valued functions which are analytic everywhere on the complex sphere $\mathbf{C}_* = \mathbf{C} \cup \{\infty\}$ outside of Λ. Passing from ζ to $\zeta = 1/\zeta$, setting

$$R_A(\zeta) = (I - \zeta A)^{-1}, \; R_{A_K}(\zeta) = (I - \zeta[A + BKC])^{-1},$$

$\Gamma_{\rm o}^+ = \{\zeta : 1/\zeta \in \Gamma_{\rm o}\}$ and taking into account (3.11), (3.12), (3.17), we can reformulate the statement to be proved equivalently as follows:

(!) *on $\Gamma_{\rm o}^+$, the function*

$$[\zeta R_{A_K}(\zeta)[BKD + R]; K[D + \zeta C R_{A_K}(\zeta)[BKD + R]]] \tag{3.18}$$

can be uniformly approximated within any desired accuracy by functions of the form

$$\left[\zeta R_A(\zeta)R + \zeta R_A(\zeta)B\left(\sum_{s=0}^{k-1}\zeta^s H_s\right)[D + \zeta C R_A(\zeta)R]\,; \left(\sum_{s=0}^{k-1}\zeta^s H_s\right)[D + \zeta C R_A(\zeta)R]\right]. \tag{3.19}$$

To prove (!), observe that $\Gamma_{\rm o}^+$ is a simply connected closed and bounded domain in $\mathbf{C}$ and that the functions $R_{A_K}(\cdot)$ and $R_A(\cdot)$ are analytic in a neighbourhood of $\Gamma_{\rm o}^+$. It follows that the function

$$\Phi(\zeta) := (I - \zeta R_A(\zeta)BKC)^{-1} = (I - \zeta(I-\zeta A)^{-1}BKC)^{-1} = (I - \zeta[A+BKC])^{-1}(I-\zeta A) \quad (3.20)$$

also is analytic in a neighbourhood of $\Gamma_{\rm o}^+$. Now, by the Sherman-Morrison formula, the matrices $I_m - PQ$ and $I_n - QP$, where $P \in \mathbf{C}^{m\times n}$, $Q \in \mathbf{C}^{n\times m}$, are singular/nonsingular simultaneously, and when they are nonsingular, one has $(I_m - PQ)^{-1} = I_m + P(I_n - QP)^{-1}Q$. It follows that the function

$$F(\zeta) \equiv (I - \zeta CR_A(\zeta)BK)^{-1}$$

is analytic in a neighbourood of $\Gamma_{\rm o}^+$ along with $\Phi(\zeta)$, and

$$\Phi(\zeta) = I + \zeta R_A(\zeta)BKF(\zeta)C. \quad (3.21)$$

Now, since the matrix-valued function $\widehat{F}(\zeta) = KF(\zeta)$ is analytic in a neighbourhood of the simply connected closed and bounded domain $\Gamma_{\rm o}^+$, this function, by the standard facts of Complex Analysis, can be approximated uniformly on $\Gamma_{\rm o}^+$ by matrix-valued polynomials, i.e., there exists a sequence

$$F_k(\zeta) = \sum_{s=0}^{k-1} \zeta^s H_s^k$$

of matrix-valued polynomials such that

$$\max_{\zeta\in\Gamma_{\rm o}^+} \|\widehat{F}(\zeta) - F_k(\zeta)\| \to 0,\ k\to\infty.$$

Consequently, with $H_s = H_s^k$, $0 \le s \le k-1$, the matrix-valued functions (3.19) converge uniformly on $\Gamma_{\rm o}^+$ as $k\to\infty$ to

$$\begin{bmatrix} \zeta R_A R + \zeta R_A BK\,(I - \zeta CR_A BK)^{-1}\,[D + \zeta CR_A R] \\ K\,(I - \zeta CR_A BK)^{-1}\,[D + \zeta CR_A R] \end{bmatrix}, \quad (3.22)$$

(we write R_A instead of $R_A(\zeta)$). It remains to show that the latter matrix-valued function is nothing but (3.18). Recalling that

$$R_{A_K}(\zeta) = \Phi(\zeta)R_A = R_A + \zeta R_A B\widehat{F}(\zeta)CR_A$$

(see (3.20), (3.21)) and substituting the expression for $\widehat{F}(\zeta)$, we get

$$R_{A_K}(\zeta) = R_A + \zeta R_A BK(I - \zeta CR_A BK)^{-1}CR_A.$$

Therefore the matrix-valued function defined by (3.18) is

$$\begin{bmatrix} \zeta[R_A + \zeta R_A BK(I-\zeta CR_A BK)^{-1}CR_A][BKD + R] \\ K\left[D + \zeta C[R_A + \zeta R_A BK(I - \zeta CR_A BK)^{-1}CR_A][BKD + R]\right] \end{bmatrix}. \quad (3.23)$$

In order to show that (3.23) is identical to (3.22) on $\Gamma_{\rm o}^+$, we should verify that on $\Gamma_{\rm o}^+$ the following equalities hold:

$$\begin{array}{ll} (a) & [I + \zeta BK(I - \zeta CR_A BK)^{-1}CR_A]BK = BK(I - \zeta CR_A BK)^{-1} \\ (b) & I + \zeta CR_A[I + \zeta BK(I - \zeta CR_A BK)^{-1}CR_A]BK = (I - \zeta CR_A BK)^{-1} \\ (c) & CR_A[I + \zeta BK(I - \zeta CR_A BK)^{-1}CR_A] = (I - \zeta CR_A BK)^{-1}CR_A. \end{array} \quad (3.24)$$

We have

$$[I+\zeta BK(I-\zeta CR_ABK)^{-1}CR_A]BK = BK[I+(I-\zeta CR_ABK)^{-1}\zeta CR_ABK]$$
$$= BK(I-\zeta CR_ABK)^{-1},$$

as required in (3.24.a). Further,

$$I+\zeta CR_A[I+\zeta BK(I-\zeta CR_ABK)^{-1}CR_A]BK$$
$$= I+\zeta CR_ABK[I+(I-\zeta CR_ABK)^{-1}\zeta CR_ABK]$$
$$= I+\zeta CR_ABK(I-\zeta CR_ABK)^{-1} = (I-\zeta CR_ABK)^{-1},$$

as required in (3.24.b). Finally,

$$CR_A[I+\zeta BK(I-\zeta CR_ABK)^{-1}CR_A] = [I+\zeta CR_ABK(I-\zeta CR_ABK)^{-1}]CR_A$$
$$= (I-\zeta CR_ABK)^{-1}CR_A,$$

as required in (3.24.c). □

4 Accounting for gains

In this section we study control problems with gain-type specifications other than H_∞-gains, and show how to reduce them to convex programs, using the bi-affinity of states and controls in (z, d^{t-1}) and in the "parameter" η of purified-output-based control law.

Let the open loop system (2.5) be "closed" by control law (1.3). The states and the controls in the closed-loop system are then affine functions of (z, d^{t-1}):

$$x_t = X_t[\eta] + X_t^z[\eta]z + X_t^d[\eta]d^{t-1},\ \ u_t = U_t[\eta] + U_t^z[\eta]z + U_t^d[\eta]d^{t-1}, \tag{4.25}$$

where $X_t[\eta], U_t[\eta]$ are vectors, and $X_z^t[\eta], X_d^t[\eta], U_z^t[\eta], U_d^t[\eta]$ are matrices ll of which affinely depend on $\eta = \{h_t, H_{t\tau}\}$. Now let us define the *gains* – the quantities

$$\begin{array}{rclrcl} z2x_t[\eta] &=& \max\limits_{\|z\|_{(z)}\leq 1} \|X_t^z[\eta]z\|_{(x)}, & z2u_t[\eta] &=& \max\limits_{\|z\|_{(z)}\leq 1} \|U_t^z[\eta]z\|_{(u)} \\ d2x_t[\eta] &=& \max\limits_{\|d^{t-1}\|_{(d)}\leq 1} \|X_t^d[\eta]d\|_{(x)}, & d2u_t[\eta] &=& \max\limits_{\|d^{t-1}\|_{(d)}\leq 1} \|U_t^d[\eta]d\|_{(u)} \end{array} \tag{4.26}$$

where $\|\cdot\|_{(z)}, ..., \|\cdot\|_{(u)}$ are given norms on the corresponding spaces. From now on we assume that these norms are such that the induced norms of linear mappings appearing in (4.26) are efficiently computable[3)]. Under this assumption, for every t the gains are efficiently computable convex function of η (due to the affinity of $X_t^z[\eta], ..., U_t^d[\eta]$ in η). It follows that design specifications expressed as upper bounds on (finitely many) gains, or on their positively weighted sums, are efficiently computable convex constraints on η.

As an instructive example, let $\mathcal{C}_{k,T}$ be the family of all *nearly time invariant purified-output-based affine control laws of order k with stabilization time T*, that is, of control laws (1.3) with

[3)]Note that the norm $\|A\|_{p\to r} = \max\limits_{\|x\|_p\leq 1}\|Ax\|_r$, $p, r\in[1,\infty]$ of a linear mapping $A\in\mathbf{R}^{m\times n}$ is efficiently computable when (a) $p=\infty$, (b) $r=1$, and (c) $p=r=2$. Besides this, in the case of $p\geq 2\geq r$, the norm $\|A\|_{p\to r}$ admits an efficiently computable convex in A upper bound tight within an absolute constant factor 2.2936...[4].

$h_t \equiv 0$, $H_{t\tau} = 0$ for $t - \tau \geq k$ and $H_{t\tau}$ depending solely on $t - \tau$ for $t \geq T$ (so that the control laws from the family become time invariant when $t \geq T$).

Let the criteria we are interested in be

(a) the four *global* gains $z2x[\eta] = \sup_{t\geq 0} z2x_t[\eta]$,...,$d2u[\eta] = \sup_{t\geq 0} d2u_t[\eta]$ corresponding to the $\|\cdot\|_\infty$-norm in the role of each of the norms $\|\cdot\|_{(z)}$,...,$\|\cdot\|_{(u)}$ (in this case all four gains are, for every t, are efficiently computable convex functions of η), and

(b) the two H_∞-gains $H_{\infty,x}[\eta] = \max_{|\zeta|=1,i,j} |(\mathcal{H}_x(\zeta))|_{ij}$, $H_{\infty,u}[\eta] = \max_{|\zeta|=1,i,j} |(\mathcal{H}_u(\zeta))|_{ij}$, where $\mathcal{H}_x$ and $\mathcal{H}_u$ are the disturbance-to-state and disturbance-to-control transfer matrices (see (3.11)). Assume that our goal is to find in a given family $\mathcal{C}_{k,T}$ a control law such that the corresponding criteria admit given in advance upper bounds (these upper bounds are our design specifications). This goal can be achieved via Convex Programming. Indeed, let us choose a "time horizon" $T_+ \geq T$ and solve the feasibility problem

$$\text{find } \eta = \{H_{t\tau}\}_{0\leq\tau<\infty} \in \mathcal{C}_{k,T} \text{ s.t. } \left\{\begin{array}{l} H_{t,\tau} = 0,\ t-\tau \geq k \\ H_{t,\tau} = H_{t',\tau'}, t, t' \geq T, t-\tau = t'-\tau' \\ H_{\infty,x}[\eta] \leq H^*_{\infty,x}, H_{\infty,u}[\eta] \leq H^*_{\infty,u} \\ z2x_t[\eta] \leq z2x^*, ..., d2u_t[\eta] \leq d2u^*, 0 \leq t \leq T_+ \end{array}\right. , \tag{4.27}$$

where $H^*_{\infty,x}, ..., d2u^*$ are the desired upper bounds on the corresponding criteria. Note that (4.27) is in fact a feasibility problem with finitely many variables $\mathcal{H}_{t\tau}$, $0 \leq \tau \leq T$, and efficiently computable constraints and thus it is computationally tractable. If the problem is infeasible, then the design specifications in question clearly are incompatible with each other. When (4.27) is feasible, we can pick a feasible solution to this problem and check whether this solution (which by construction satisfies the specifications on its H_∞-performance and on the "finite horizon" gains $\max_{0\leq t\leq T_+} z2x_t[\eta]$,..., $\max_{0\leq t\leq T_+} d2u_t[\eta]$) meets the specifications on the global gains $z2x[\cdot], ..., d2u[\cdot]$ as well. With properly chosen T_+, there are good chances that this indeed will be the case, otherwise, we can repeat this procedure for a larger value of T_+ and so on, terminating with a desired control law (or the conclusion that no such law exists), provided, of course, that termination occurs before problems (4.27) become too large for numerical processing.

5 Numerical illustration

In this section, we present a sample of small examples illustrating the potential of the proposed approach.

5.1 Example 1: Discrete time H_∞-synthesis

We consider a 3-echelon supply chain comprised of 3 warehouses. The external demand is supplied from the stock at warehouse # 1; the stock at warehouse # i is replenished from warehouse # $i+1$, where "warehouse # 4" is a factory with infinite supply capacity. There exists a delay of 2 time units in executing replenishment orders. The inventory can be modelled by a 9-state discrete time linear time invariant system

$$\begin{bmatrix} x^1_{t+1} \\ x^2_{t+1} \\ x^3_{t+1} \\ x^4_{t+1} \\ x^5_{t+1} \\ x^6_{t+1} \\ x^7_{t+1} \\ x^8_{t+1} \\ x^9_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1&0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0 \end{bmatrix}}_{A_0} \begin{bmatrix} x^1_t \\ x^2_t \\ x^3_t \\ x^4_t \\ x^5_t \\ x^6_t \\ x^7_t \\ x^8_t \\ x^9_t \end{bmatrix} + \begin{bmatrix} 0&0&0 \\ -1&0&0 \\ 0&-1&0 \\ 0&0&0 \\ 1&0&0 \\ 0&0&0 \\ 0&1&0 \\ 0&0&0 \\ 0&0&1 \end{bmatrix} \begin{bmatrix} w^1_t \\ w^2_t \\ w^3_t \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d_t, \qquad (5.28)$$

where x^1_t, x^2_t, x^3_t are levels of inventories at warehouses 1, 2, 3 at time t, w^1_t, w^2_t, w^3_t are replenishment orders issued at time t by the respective warehouses, d_t is the external demand at time t and $x^4_t, ..., x^9_t$ are analysis variables. In our experiment, we assume that the outputs are the inventory levels: $y_t = (x^1_t, x^2_t, x^3_t)^T$. Matrix A_0 is "neutral": its eigenvalues are 1 (multiplicity 3, three 1×1 cells in the Jordan canonical form) and 0 (multiplicity 6, three 2×2 cells in the Jordan canonical form). We start with finding an output-based linear feedback $u_t = Ky_t$ which stabilizes the closed loop system. For the toy sizes of the example, such a feedback can be found by straightforward random sampling of 3×3 matrices K until a stabilizing feedback is found. This most naive procedure in our experiment yielded the feedback

$$K = \begin{bmatrix} -1.0135 & 0.2766 & 0.1059 \\ -1.7559 & -0.0140 & 0.5248 \\ -1.1766 & 0.1833 & 0.0528 \end{bmatrix},$$

resulting in the maximum of modulae of the eigenvalues of $A = A_0 + BKC$ about 0.951. We now can pass from the original system to the stable system

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& Ax_t + Bu_t + Rd_t \ . \\ y_t &=& (x^1_t, x^2_t, x^3_t)^T \end{array} \qquad (5.29)$$

"Closing" system (5.29) with a time invariant purified-output-based control law (2.6) and "translating" this law to the original system (5.28), we end up with the closed loop system given by

$$\begin{array}{rcl} x_0 &=& z \\ x_{t+1} &=& A_0x_t + B(KCx_t + u_t) + Rd_t \\ y_t &=& Cx_t \\ \hline \widehat{x}_0 &=& 0 \\ \widehat{x}_{t+1} &=& [A_0 + BKC]\widehat{x}_t + Bu_t \\ \widehat{y}_t &=& C\widehat{x}_t \\ \hline v_t &=& y_t - \widehat{y}_t \\ \hline u_t &=& \sum\limits_{s=0}^{k-1} H_s v_{t-s} \end{array} \qquad (5.30)$$

(recall that by our convention all entities with negative indices vanish). By the preceding analysis, with harmonic oscillation $d_t = \exp\{\imath\omega t\}d$ in the role of external disturbance, the "state-control"

k	0	1	2	4	8
$\phi_x(\omega)$	23.074	5.206	2.795	2.449	2.379

Table 1: Peak state frequency response vs. the order k of the optimal time invariant purified-output-based affine control law. $k = 0$ corresponds to "no control": in (5.30), $u_t \equiv 0$.

component $[x_t; u_t]$ of every solution to (5.30) approaches, as $t \to \infty$, the steady-state harmonic oscillation

$$[\widetilde{x}_t; \widetilde{u}_t] = \exp\{\imath\omega t\}[\mathcal{H}_x(\exp\{\imath\omega\}); \mathcal{H}_u(\exp\{\imath\omega\})]d$$

with $\mathcal{H}_x(z)$, $\mathcal{H}_u(z)$ given by (3.11). Since the actual controls w_t in (5.28) are linked to the states x_t and controls w_t according to $w_t = KCx_t + u_t$, with our affine control law and $d_t = \exp\{\imath\omega t\}d$ the actual state-control trajectory $[x_t; w_t]$ of (5.28) approaches, as $t \to \infty$, the harmonic oscillation

$$[\widetilde{x}_t; \widetilde{w}_t] = \exp\{\imath\omega t\}\mathcal{H}_{xw}(\exp\{\imath\omega\})d, \ \mathcal{H}_{xw}(z) = [\mathcal{H}_x(z); \mathcal{H}_w(z)] \equiv [\mathcal{H}_x(z); \mathcal{H}_u(z) + KC\mathcal{H}_x(z)].$$

The resulting transfer function $\mathcal{H}_{xw}(\cdot)$ is affine in the parameters $H = [H_0, H_1, ..., H_{k-1}]$ of the underlying affine control law and is a rational matrix-valued function without singularities outside of the open unit disk. Thus, all nice already outlined consequences of the affinity of the transfer function in H are valid in this example.

Let $\alpha_\ell(\omega)$ be the amplitude of the steady-state oscillation of state x^ℓ caused by a unit amplitude disturbing harmonic oscillation $d_t = \exp\{\imath\omega t\}$ of frequency ω. We refer to the function $\phi_x(\omega) = \max\limits_\ell \alpha_\ell(\omega)$ as to *state frequency response*. The control frequency response $\phi_u(\omega)$ is defined similarly, with the steady state oscillations of controls w^ℓ in the role of steady state oscillations of states. In our experiment, we were interested to minimize the maximum, over $\omega \in [0, 2\pi]$, of the state frequency response. This leads to solving the semi-infinite convex optimization program

$$\min_{\tau,H} \{\tau : \tau \geq \|\mathcal{H}_x(\exp\{\imath\omega\})\|_\infty \ \ \forall\omega \in [0, 2\pi]\} \tag{5.31}$$

(note that our disturbances are scalar, so that the values of $\mathcal{H}_x(\cdot)$ as given by (3.11) are column vectors of dimension 9). As it was explained in Section 3, this semi-infinite problem can be reduced straightforwardly to an explicit semidefinite program. In our experiment, however, we did not use this possibility (which would require to work with high-dimensional SDP's) and merely replaced the domain $[0, 2\pi]$ of ω with a "fine finite grid" Γ of values of ω, thus approximating (5.31) by the conic quadratic program

$$\min_{\tau,H} \{\tau : \tau \geq \|\mathcal{H}_x(\exp\{\imath\omega\})\|_\infty \, \forall\omega \in \Gamma\}. \tag{5.32}$$

Note that restricting the "design parameters" H to be real (which is the only meaningful case in the inventory context), we enforce $\mathcal{H}(z)$ to be "symmetric": $H_\ell(z^*) = (H_\ell(z))^*$. It follows that we lose nothing when choosing grid Γ in $[0, \pi]$ rather than in $[0, 2\pi]$. In our experiment, Γ was chosen as the 128-point equidistant grid on $[0, \pi]$; we solved (5.32) for $k = 1, 2, 4, 8$ and then measured the quality of the resulting solution (the quantity $\max\limits_{0\leq\omega\leq 2\pi} \|\mathcal{H}_x(\exp\{\imath\omega\})\|_\infty$) by computing $\|\mathcal{H}_x(\exp\{\imath\omega\})\|_\infty$ along a 4096-point equidistant grid of values of ω. The results of our experiment are summarized in Table 1 and depicted on Figure 1.

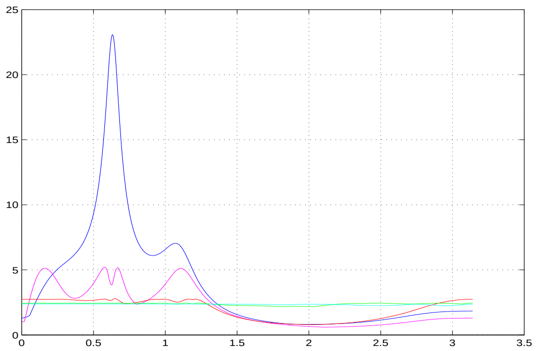


Figure 1: State frequency response vs. $\omega \in [0, \pi]$ for the optimal time invariant purified-output-based affine control law of order k. Blue: $k = 0$; magenta: $k = 1$; red: $k = 2$; green: $k = 4$; cyan: $k = 8$.

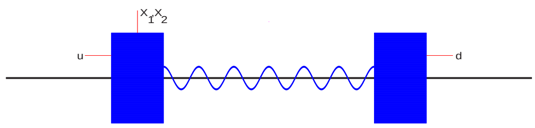


Figure 2: Double pendulum: two masses linked by spring sliding without friction along a rode. Position and velocity of the first mass are observed.

5.2 Example 2: Discrete time L_1-synthesis

The open-loop system we intend to consider now is the (discretized) double-pendulum depicted on Fig. 2. The dynamics of the continuous time prototype plant is given by

$$\begin{array}{rcl} \dot{x} & = & = A_c x + B_c u + R_c d \\ y & = & Cx \end{array}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, R_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(x_1, x_2 are the position and the velocity of the first mass, x_3, x_4 – those of the second mass). The discrete time plant we will actually work with is

$$\begin{array}{rcl} x_{t+1} & = & A_0 x_t + B u_t + R d_t \\ y_t & = & C x_t \end{array} \qquad (5.33)$$

where $A_0 = \exp\{\Delta \cdot A_c\}$, $B = \int_0^\Delta \exp\{sA_c\} B_c ds$, $R = \int_0^\Delta \exp\{sA_c\} R_c ds$. In our experiment, we used $\Delta = 0.2500$ ($\approx 1/18$-th of the period of pendulum's oscillating modes). Same as in Example 1, system (5.33) is not stable (modulae of all eigenvalues of A_0 are equal to 1), and we start with picking, again by brute force random search, an output-based linear time invariant feedback which stabilizes the system. Our search yielded feedback matrix $K = [-0.6950, -1.7831]$, with the spectral radius of the matrix $A = A_0 + BKC$ equal to 0.87. From now on, we focus on the stabilized version of (5.33), that is, on the discrete time open loop system

$$\begin{array}{rcl} x_{t+1} & = & Ax_t + Bw_t + Rd_t \\ y_t & = & Cx_t \end{array} \tag{5.34}$$

which we intend to "close" by a control law from $\mathcal{C}_{8,0}$, that is, by time invariant control given by

$$w_t = \sum_{\tau=0}^{t} H_{t-\tau} v_\tau \quad \left[\begin{array}{rcl} v_t & = & y_t - C\widehat{x}_t, \\ \widehat{x}_{t+1} & = & A\widehat{x}_t + Bw_t,\ \widehat{x}_0 = 0 \end{array} \right] \tag{5.35}$$

where $H_s = 0$ when $s \geq 8$. Our goal is to pick in $\mathcal{C}_{8,0}$ a control law with desired properties (to be precisely specified a bit later) expressed in terms of the 6 criteria $z2x$,...,$H_{\infty,u}$ defined in Section 4. Note that while the purified-output-based control we are seeking is defined in terms of the stabilized plant (5.34), the criteria $z2u$,$d2u$, $H_{\infty,u}$ are defined in terms of the controls $u_t = w_t + BKCx_t$ affecting the actual plant (5.33).

In the synthesis we are about to describe, our primary goal is to minimize the global disturbance-to-state gain $d2x$, while the secondary goal is to avoid too large values of the remaining criteria. We achieve this goal as follows.

Step 1: Optimizing $d2x$. As it was explained in Section 4, the optimization problem

$$\mathrm{Opt}_{d2x}(k, 0; T_+) = \min_{\eta \in \mathcal{C}_{k,0}} \max_{0 \leq t \leq T_+} d2x_t[\eta] \tag{5.36}$$

is an explicit convex program (in fact, just a linear programming one), and its optimal value is a lower bound on the best possible global gain $d2x$ achievable with control laws from $\mathcal{C}_{k,0}$. In our experiment, we solve (5.36) for $k = 8$ and $T_+ = 40$, arriving at $\mathrm{Opt}_{d2x}(8, 0; 40) = 1.773$. The global $d2x$-gain of the resulting time-invariant control law is 1.836 – just by 3.5% larger than the outlined lower bound. We conclude that the control yielded by the solution to (5.36) is nearly the best one, in terms of the global $d2x$-gain, among time-invariant controls of order 8. At the same time, part of the other gains associated with this control are far from being good, see line "$d2x^{40}$" in Table 2.

Step 2: Improving the remaining gains. To improve the "bad" gains yielded by the nearly $d2x$-optimal control law we have built, we act as follows: we look at the family $\mathcal{F}$ of all time invariant control laws of order 8 with the finite-horizon $d2x$-gain $d2x^{40}[\eta] = \max\limits_{0 \leq t \leq 40} d2x_t[\eta]$ not exceeding 1.90 (that is, the controls from $\mathcal{C}_{8,0}$ which are within 7.1% of the optimum in terms of their $d2x^{40}$-gain) and act as follows:

A. We optimize over $\mathcal{F}$, one at a time, every one of the remaining criteria $z2x^{40}[\eta] = \max\limits_{0 \leq t \leq 40} z2x_t[\eta]$, $z2u^{40}[\eta] = \max\limits_{0 \leq t \leq 40} z2u_t[\eta]$, $d2u^{40}[\eta] = \max\limits_{0 \leq t \leq 40} d2u_t[\eta]$, $H_{\infty,x}[\eta]$, $H_{\infty,u}[\eta]$, thus obtaining "reference values" of these criteria; these are lower bounds on the optimal values of the corresponding global gains, optimization being carried out over the set $\mathcal{F}$. These lower bounds are the underlined data in Table 2.

Optimized criterion	Resulting values of the criteria					
	$z2x^{40}$	$z2u^{40}$	$d2x^{40}$	$d2u^{40}$	$H_{\infty,x}$	$H_{\infty,u}$
$z2x^{40}$	$\underline{25.8}$	205.8	1.90	3.75	10.52	5.87
$z2u^{40}$	58.90	$\underline{161.3}$	1.90	3.74	39.87	20.50
$d2x^{40}$	5773.1	13718.2	$\underline{1.77}$	6.83	1.72	4.60
$d2u^{40}$	1211.1	4903.7	1.90	$\underline{2.46}$	66.86	33.67
$H_{\infty,x}$	121.1	501.6	1.90	5.21	$\underline{1.64}$	5.14
$H_{\infty,u}$	112.8	460.4	1.90	4.14	8.13	$\underline{1.48}$
	$z2x$	$z2u$	$d2x$	$d2u$	$H_{\infty,x}$	$H_{\infty,u}$
(5.37)	31.59	197.75	1.91	4.09	1.82	2.04
(5.38)	2.58	0.90	1.91	4.17	1.77	1.63

Table 2: Gains for time invariant control laws of order 8 yielded by optimizing, one at a time, the criteria $z2x^{40}$,...,$H_{\infty,u}$ over control laws from $\mathcal{F} = \{\eta \in \mathcal{C}_{8,0} : d2x^{40}[\eta] \leq 1.90\}$ (first six lines), and by solving programs (5.37), (5.38) (last two lines).

B. We then minimize over $\mathcal{F}$ the "aggregated gain"

$$\frac{z2x^{40}[\eta]}{25.8} + \frac{z2u^{40}[\eta]}{161.3} + \frac{d2u^{40}[\eta]}{2.46} + \frac{H_{\infty,x}[\eta]}{1.64} + \frac{H_{\infty,u}[\eta]}{1.48} \tag{5.37}$$

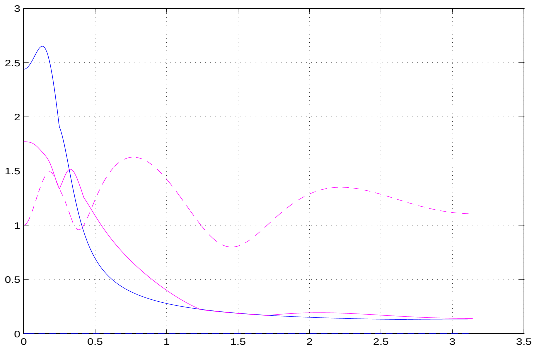
(the denominators are exactly the aforementioned reference values of the corresponding gains). The global gains of the resulting time-invariant control law of order 8 are presented in the "(5.37)" line of Table 2.

Step 3: Finite-horizon adjustments. Our last step is to improve the $z2x$- and $z2u$-gains by passing from a time invariant affine control law of order 8 to a nearly time invariant law of order 8 with stabilization time $T = 20$. To this end, we solve the convex optimization problem

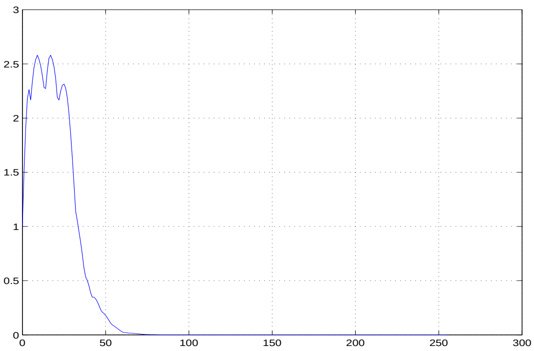
$$\min_{\eta \in \mathcal{C}_{8,20}} \left\{ z2x^{50}[\eta] + z2u^{50}[\eta] : \begin{array}{rcl} d2x^{50}[\eta] & \leq & 1.90 \\ d2u^{50}[\eta] & \leq & 4.20 \\ H_{\infty,x}[\eta] & \leq & 1.87 \\ H_{\infty,u}[\eta] & \leq & 2.09 \end{array} \right\} \tag{5.38}$$

(the right hand sides in the constraints for $d2u^{50}[\cdot]$, $H_{\infty,x}[\cdot]$, $H_{\infty,u}[\cdot]$ are the slightly increased (by 2.5%) gains of the time invariant control law obtained at Step 2). The global gains of the resulting control law are presented in the last line of Table 2, see also Fig. 3. We see that finite-horizon adjustments allow to reduce by orders of magnitude the global $z2x$- and $z2u$-gains and, as an additional bonus, result in a substantial reduction of H_∞-gains.

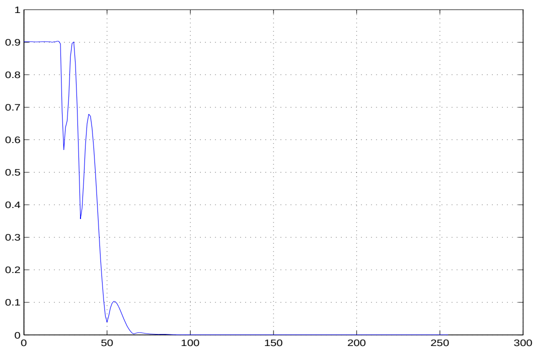
Simple as this control problem may be, it serves well to demonstrate the importance of purified-output-based representation of affine control laws and the associated possibility to express various control specifications as explicit convex constraints on the parameters of such a law.



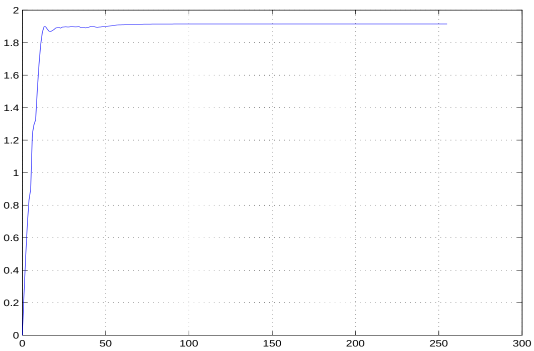
State and control frequency responses. Blue: no control; magenta, solid: states; magenta, dashed: controls.



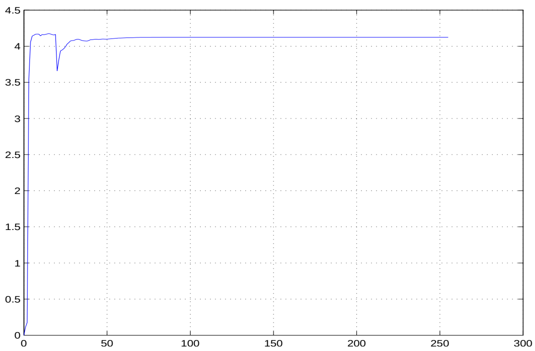
$z2x_t$, $0 \leq t \leq 256$



$z2u_t$, $0 \leq t \leq 256$



$d2x_t$, $0 \leq t \leq 256$



$d2u_t$, $0 \leq t \leq 256$

Figure 3: Frequency responses and gains of control law given by solution to (5.38).

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