Robust Minimum Variance Beamforming

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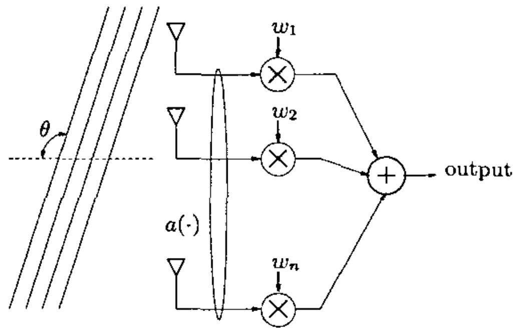

Fig. 1. Beamformer block diagram.

Abstract— **This paper introduces an extension of minimum variance beamforming that explicitly uses the a-priori uncertainty in the array response. Sources of this uncertainty include imprecise knowledge of the angle of arrival and uncertainty in the array manifold; this uncertainty is modeled via an ellipsoid. We choose weights that minimize the total weighted power output of the array, subject to the constraint that the gain exceeds unity for all possible array responses in this ellipsoid. We show that the robust weights can be computed efficiently using Lagrange multiplier techniques.**

Numerical examples are presented.

I. Introduction

CONSIDER an array of n sensors. Let $a(\theta) \in \mathbf{C}^n$ denote the response of the array to a plane wave of unit amplitude arriving from direction θ; we shall refer to $a(\cdot)$ as the *array manifold.* We assume that a narrow-band source $s(t)$ is impinging upon the array from angle θ and that the source is in the far-field of the array. The vector array output $y(t) \in \mathbf{C}^n$ is then:

$$y(t) = a(\theta)s(t) + v(t), \qquad (1)$$

where $a(\theta)$ includes effects such as coupling between elements and subsequent amplification; $v(t)$ is a vector of additive noises representing the effect of undesired signals, such as thermal noise or interference. We denote the sampled array output by $y(k)$. Similarly, the combined beamformer output is given by

$$y_c(k) = w^*y(k) = w^*a(\theta)s(k) + w^*v(k),$$

where $w \in \mathbf{C}^n$ is a vector of weights, *i.e.*, design variables, and $(\cdot)^*$ denotes the conjugate transpose.

The goal is to make $w^*a(\theta) \approx 1$ and $w^*v(t)$ small, in which case, $y_c(t)$ recovers $s(t)$, *i.e.*, $y_c(t) \approx s(t)$. The gain of the weighted array response in direction θ is $|w^*a(\theta)|$; the expected effect of the noise and interferences at the combined output is given by w^*R_vw, where $R_v = \mathbf{E}\,vv^*$ and $\mathbf{E}$ denotes the expected value. If we presume $a(\theta)$ and R_v are known, we may choose w as the optimal solution of

$$\begin{array}{ll} \text{minimize} & w^*R_vw \\ \text{subject to} & w^*a(\theta_d) = 1. \end{array} \qquad (2)$$

Minimum variance beamforming is a variation on (2) in which we replace R_v with an estimate of the received signal covariance derived from the sample covariance of recently received samples of the array output, *e.g.*,

$$R_y(k) = \frac{1}{N}\sum_{i=k-N+1}^{k} y(i)y(i)^* \in \mathbf{C}^{n\times n}. \qquad (3)$$

The minimum variance beamformer (MVB) is chosen as the optimal solution of

$$\begin{array}{ll} \text{minimize} & w^*R_yw \\ \text{subject to} & w^*a(\theta) = 1. \end{array} \qquad (4)$$

This is commonly referred to as Capon's method [1]. Equation (4) has an analytical solution given by

$$w_{\text{mv}} = \frac{R_y^{-1}a(\theta)}{a(\theta)^*R_y^{-1}a(\theta)}. \qquad (5)$$

Equation (4) also differs from (2) in that the power expression we are minimizing includes the effect of the desired signal plus noise. The constraint $w^*a(\theta) = 1$ in (4) prevents the gain in the direction of the signal from being reduced.

A measure of the effectiveness of a beamformer is given by the signal to interference plus noise ratio, commonly abbreviated as SINR, given by

$$\text{SINR} = \frac{\sigma_d^2|w^*a(\theta)|^2}{w^*R_vw}, \qquad (6)$$

where σ_d^2 is the power of the signal of interest. The assumed value of the array manifold $a(\theta)$ may differ from the actual value for a host of reasons including imprecise knowledge of the signal's angle of arrival θ. Unfortunately, the SINR of Capon's method can degrade catastrophically for modest differences between the assumed and actual values of the array manifold. We now review several techniques for minimizing the sensitivity of MVB to modeling errors in the array manifold.

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A. Previous work

One popular method to address uncertainty in the array response or angle of arrival is to impose a set of unity-gain constraints for a small spread of angles around the nominal look direction. These are known in the literature as point mainbeam constraints or neighboring location constraints [2]. The beamforming problem with point mainbeam constraints can be expressed as

$$\begin{array}{ll} \text{minimize} & w^* R_y w \\ \text{subject to} & C^* w = f, \end{array} \tag{7}$$

where C is a $n \times L$ matrix of array responses in the L constrained directions and f is an $L \times 1$ vector specifying the desired response in each constrained direction. To achieve wider responses, additional constraint points may be added. We may similarly constrain the derivative of the weighted array output to be zero at the desired look angle. This constraint can be expressed in the same framework as (7); in this case, we let C be the derivative of the array manifold with respect to look angle and $f = 0$. These are called *derivative mainbeam constraints;* this derivative may be approximated using regularization methods. Point and derivative mainbeam constraints may also be used in conjunction with one another. The minimizer of (7) has an analytical solution given by:

$$w_{\text{opt}} = R_y^{-1} C (C^* R_y^{-1} C)^{-1} f. \tag{8}$$

Each constraint removes one of the remaining degrees of freedom available to reject undesired signals; this is particularly significant for an array with a small number of elements. We may overcome this limitation by using a low-rank approximation to the constraints [3]. The best rank-k approximation to C, in a least squares sense, is given by $U \Sigma V^*$, where Σ is a diagonal matrix consisting of the largest k singular values, U is a $n \times k$ matrix whose columns are the corresponding left singular vectors of C, and V is a $L \times k$ matrix whose columns are the corresponding right singular vectors of C. The reduced rank constraint equations can be written as $V \Sigma^T U^* w = f$, or equivalently:

$$U^* w = \Sigma^\dagger V^* f, \tag{9}$$

where $\dagger$ denotes the Moore-Penrose pseudoinverse. Using (8), we compute the beamformer using the reduced rank constraints as

$$w_{\text{epc}} = R_y^{-1} U (U^* R_y^{-1} U)^{-1} \Sigma^\dagger V^* f.$$

This technique, used in source localization, is referred to as minimum variance beamforming with environmental perturbation constraints (MV-EPC), see [2] and the references contained therein.

Unfortunately, it is not clear how best to pick the additional constraints, or, in the case of the MV-EPC, the rank of the constraints. The effect of additional constraints on the design specifications appears difficult to predict.

Regularization methods have also been used in beamforming. One technique, referred to in the literature as diagonal loading, chooses the beamformer to minimize the sum of the weighted array output power plus a penalty term, proportional to the square of the norm of the weight vector. The gain in the assumed AOA of the desired signal is constrained to be unity. The beamformer is chosen as the optimal solution of:

$$\begin{array}{ll} \text{minimize} & w^* R_y w + \mu w^* w \\ \text{subject to} & w^* a(\theta) = 1. \end{array} \tag{10}$$

The parameter $\mu > 0$ penalizes large values of w and has the general effect of *detuning* the beamformer response. The regularized least squares problem (10) has an analytical solution given by:

$$w_{\text{reg}} = \frac{(R_y + \mu I)^{-1} a(\theta)}{a(\theta)^* (R_y + \mu I)^{-1} a(\theta)}. \tag{11}$$

Gershman [4] and Johnson and Dudgeon [5] provide a survey of these methods; see also the references contained therein. Similar ideas have been used in adaptive algorithms, see [6].

Beamformers using eigenvalue thresholding methods to achieve robustness have also been used; see [7]. The beamformer is computed according to Capon's method using a covariance matrix which has been modified to ensure no eigenvalue is less than a factor μ times the largest, where $0 \leq \mu \leq 1$. The performance of this beamformer appears similar to that of the regularized beamformer using diagonal loading; both usually work well for an appropriate choice of the regularization parameter μ.

We see two limitations with regularization techniques for beamformers. First, it is not clear how to efficiently pick μ. Second, this technique does not take into account any knowledge we may have about variation in the array manifold, *e.g.*, that the variation may not be isotropic.

In §I-C, we describe a beamforming method that explicitly uses information about the variation in the array response $a(\cdot)$, which we model explicitly as an uncertainty ellipsoid. Prior to this, we introduce some notation for describing ellipsoids.

B. Ellipsoid descriptions

A n-dimensional ellipsoid can be defined as the image of a n-dimensional Euclidean ball under an affine mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$, *i.e.*,

$$\mathcal{E} = \{Au + c \mid \|u\| \leq 1\}, \tag{12}$$

where $A \in \mathbf{R}^{n \times n}$ and $c \in \mathbf{R}^n$. The set $\mathcal{E}$ describes an ellipsoid whose center is c and whose *principle semiaxes* are the unit-norm left singular vectors of A scaled by the corresponding singular values. We say that an ellipsoid is *flat* if this mapping is not injective, *i.e.*, one-to-one. Flat ellipsoids can be described by (12) in the proper affine subspaces of $\mathbf{R}^n$. In this case, $A \in \mathbf{R}^{n \times l}$ and $u \in \mathbf{R}^l$ with $n < l$.

Unless otherwise specified, an ellipsoid in $\mathbf{R}^n$ will be parameterized in terms of its center $c \in \mathbf{R}^n$ and a symmetric non-negative definite configuration matrix $P \in \mathbf{R}^{n \times n}$ as

$$\mathcal{E}(c, P) = \{P^{1/2} u + c \mid \|u\| \leq 1\} \tag{13}$$

where $P^{1/2}$ is any matrix square root satisfying $P^{1/2}(P^{1/2})^T = P$. When P is full rank, the non-degenerate ellipsoid $\mathcal{E}(c, P)$ may also be expressed as

$$\mathcal{E}(c,P) = \{x \mid (x-c)^T P^{-1}(x-c) \leq 1\}. \quad (14)$$

The first representation (13) is more natural when $\mathcal{E}$ is degenerate or poorly conditioned. Using the second description (14), one may quickly determine whether a point is within the ellipsoid.

We will express the values of the array manifold $a \in \mathbf{C}^n$ as the direct sum of its real and imaginary components in $\mathbf{R}^{2n}$; *i.e.*,

$$z_i = [\,\mathbf{Re}(a_1) \cdots \mathbf{Re}(a_n)\ \mathbf{Im}(a_1) \cdots \mathbf{Im}(a_n)]^T. \quad (15)$$

While it is possible to cover the field of values with a complex ellipsoid in $\mathbf{C}^n$, doing so implies a symmetry between the real and imaginary components which generally results in a larger ellipsoid than if the direct sum of the real and imaginary components are covered in $\mathbf{R}^{2n}$.

C. Robust minimum variance beamforming

A generalization of (4) that captures our desire to minimize the weighted power output of the array in the presence of uncertainties in $a(\theta)$ is given by:

$$\begin{array}{ll} \text{minimize} & w^* R_y w \\ \text{subject to} & \mathbf{Re}\ w^* a \geq 1 \quad \forall a \in \mathcal{E}, \end{array} \quad (16)$$

where $\mathbf{Re}$ denotes the real part. Here, $\mathcal{E}$ is an ellipsoid that covers the possible range of values of $a(\theta)$ due to imprecise knowledge of the array manifold $a(\cdot)$, uncertainty in the angle of arrival θ, or other factors. We shall refer to the optimal solution of (16) as the robust minimum variance beamformer (RMVB).

We use the constraint $\mathbf{Re}\ w^* a \geq 1$ for all $a \in \mathcal{E}$ in (16) for two reasons. First, while normally considered a semi-infinite constraint, we show in §II that it can be expressed as a second-order cone constraint. As a result, the robust minimum variance beamforming problem (16) can be solved efficiently. Second, the real part of the response is an efficient lower bound for the magnitude of the response, as the objective $w^* R_y w$ is unchanged if the weight vector w is multiplied by an arbitrary shift $e^{j\phi}$. This is particularly true when the uncertainty in the array response is relatively small. It is unnecessary to constrain the imaginary part of the response to be nominally zero. The same rotation which maximizes the real part for a given level of $w^* R w$ simultaneously minimizes the imaginary component of the response.

Our approach differs from the previously mentioned beamforming techniques in that the weight selection uses the a-priori uncertainties in the array manifold in a precise way. The magnitude of the RMVB response is guaranteed to be greater than unity for all values of the array manifold in the uncertainty ellipsoid $\mathcal{E}$. Hence, an estimate of the power of the desired signal σ_d^2, is given by the weighted power out of the array, namely

$$\hat{\sigma}_d^2 = w^* R_y w, \quad (17)$$

where the sample covariance R_y is computed as in (3).

Wu and Zhang [8] observe that the array manifold may be described as a polyhedron and that the robust beamforming problem can be cast as a quadratic program. While the polyhedron approach is less conservative, the size of the description and hence the complexity of solving the problem grows with the number of vertices. Vorobyov et al. [9],[10] have described the use of second-order cone programming for robust beamforming in the case where the uncertainty in the array response is isotropic. In this paper, we consider the case in which the uncertainty is anisotropic [11], [12], [13]. We also show how this problem can be solved efficiently in practice. Prior to publication, we learned of a work similar to ours by Li et. al [14].

D. Outline of the paper

The rest of this paper is organized as follows. In §II we discuss the RMVB. A numerically efficient technique based on Lagrange multiplier methods is described; in fact, RMVB can be computed with the same order of complexity as its non-robust counterpart. A numerical example is given in §III. Our conclusions are given in §II.

II. Robust weight selection

For purposes of computation, we will express the weight vector w and the values of the array manifold a as the direct sum of the corresponding real and imaginary components

$$x = \begin{bmatrix} \mathbf{Re}\, w \\ \mathbf{Im}\, w \end{bmatrix} \qquad z = \begin{bmatrix} \mathbf{Re}\, a \\ \mathbf{Im}\, a \end{bmatrix}. \quad (18)$$

The real component of the product $w^* a$ can be written as $x^T z$; the quadratic form $w^* R_y w$ may be expressed in terms of x as $x^T R x$, where

$$R = \begin{bmatrix} \mathbf{Re}\, R_y & -\mathbf{Im}\, R_y \\ \mathbf{Im}\, R_y & \mathbf{Re}\, R_y \end{bmatrix}.$$

We will assume R is positive definite.

Let $\mathcal{E} = \{Au + c \mid \|u\| \leq 1\}$ be an ellipsoid covering the possible values of x, *i.e.*, the real and imaginary components of a. The ellipsoid $\mathcal{E}$ is centered at c; the matrix A determines its size and shape. The constraint $\mathbf{Re}\ w^* a \geq 1$ for all $a \in \mathcal{E}$ in (16) can be expressed

$$x^T z \geq 1 \quad \forall z \in \mathcal{E}, \quad (19)$$

which is equivalent to

$$-u^T A^T x \leq c^T x - 1 \ \text{ for all } \ u \ \text{ s.t. } \ \|u\| \leq 1. \quad (20)$$

Now, (20) holds for all $\|u\| \leq 1$ if and only if it holds for the value of u that maximizes $u^T A^T x$, namely $u = -\frac{A^T x}{\|A^T x\|}$. By the Cauchy-Schwartz inequality, we see that (19) is equivalent to the constraint

$$\|A^T x\| \leq c^T x - 1, \quad (21)$$

which is called a *second-order cone constraint* [15]. We can then express the robust minimum variance beamforming problem (16) as

$$\begin{array}{ll} \text{minimize} & x^T R x \\ \text{subject to} & \|A^T x\| \leq c^T x - 1, \end{array} \quad (22)$$

which is a second-order cone program. See [15], [16], [17], and [18]. The subject of robust convex optimization is covered in [19], [20], [21], [22],and [23].

By assumption, R is positive definite and the constraint $\|A^T x\| \leq c^T x - 1$ in (22) precludes the trivial minimizer of $x^T Rx$. Hence, this constraint will be tight for any optimal solution and we may express (22) in terms of real-valued quantities as

$$\begin{array}{ll} \text{minimize} & x^T Rx \\ \text{subject to} & c^T x = 1 + \|A^T x\|. \end{array} \quad (23)$$

In the case of no uncertainty where $\mathcal{E}$ is a singleton whose center is $c = [\mathbf{Re}\, a(\theta_d)^T \;\; \mathbf{Im}\, a(\theta_d)^T]^T$, (23) reduces to Capon's method and admits an analytical solution given by the MVB (5). Compared to the MVB, the RMVB adds a margin which scales with the size of the uncertainty. In the case of an isotropic array uncertainty, the optimal solution of (16) yields the same weight vector (to a scale factor) as the regularized beamformer for the proper the proper choice of μ.

A. Lagrange multiplier methods

It is natural to suspect that we may compute the RMVB efficiently using Lagrange multiplier methods. See, for example, [24], [25] [16], [26], [27, §12.1.1], and [28]. Indeed this is the case.

The RMVB is the optimal solution of

$$\begin{array}{ll} \text{minimize} & x^T Rx \\ \text{subject to} & \|A^T x\|^2 = (c^T x - 1)^2 \end{array} \quad (24)$$

if we impose the additional constraint that $c^T x \geq 1$. We define the *Lagrangian* $L : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ associated with (24) as:

$$\begin{aligned} L(x, \lambda) &= x^T Rx + \lambda(\|A^T x\|^2 - (c^T x - 1)^2) \\ &= x^T (R + \lambda Q)x + 2\lambda c^T x - \lambda, \end{aligned} \quad (25)$$

where $Q = AA^T - cc^T$. To calculate the stationary points, we differentiate $L(x, y)$ with respect to x and λ; setting these partial derivatives equal to zero we have respectively:

$$(R + \lambda Q)x = -\lambda c \quad (26)$$

and

$$x^T Qx + 2c^T x - 1 = 0, \quad (27)$$

which are known as the *Lagrange* equations. To solve for the Lagrange multiplier λ, we note that equation (26) has an analytical solution given by

$$x = -\lambda(R + \lambda Q)^{-1} c;$$

applying this to (27) yields

$$\begin{aligned} f(\lambda) \;=\; & \lambda^2 c^T (R + \lambda Q)^{-1} Q (R + \lambda Q)^{-1} c \\ & -2\lambda c^T (R + \lambda Q)^{-1} c - 1. \end{aligned} \quad (28)$$

The optimal value of the Lagrange multiplier λ^* is then a zero of (28).

We proceed by computing the eigenvalue/eigenvector decomposition

$$V \Gamma V^T = R^{-1/2} Q (R^{-1/2})^T$$

to diagonalize (28), *i.e.*,

$$\begin{aligned} f(\lambda) \;=\; & \lambda^2 \bar{c}^T (I + \lambda\Gamma)^{-1} \Gamma (I + \lambda\Gamma)^{-1} \bar{c} \\ & -2\lambda \bar{c}^T (I + \lambda\Gamma)^{-1} \bar{c} - 1, \end{aligned} \quad (29)$$

where $\bar{c} = V^T R^{-1/2} c$. Equation (29) reduces to the following scalar *secular* equation

$$f(\lambda) = \lambda^2 \sum_{i=1}^{n} \frac{\bar{c}_i^2 \gamma_i}{(1 + \lambda\gamma_i)^2} - 2\lambda \sum_{i=1}^{n} \frac{\bar{c}_i^2}{(1 + \lambda\gamma_i)} - 1, \quad (30)$$

where $\gamma \in \mathbf{R}^n$ are the diagonal elements of Γ. The values of γ are known as the *generalized eigenvalues* of Q and R and are the roots of the equation $\det(Q - \gamma R) = 0$; Having computed the value of λ^* satisfying $f(\lambda^*) = 0$, the RMVB is computed according to:

$$x^* = -\lambda^* (R + \lambda^* Q)^{-1} c. \quad (31)$$

Similar techniques have been used in the design of filters for radar applications; see Stutt and Spafford [29] and Abramovich and Sverdlik [30].

In principle, we could solve for all the roots of (30) and choose the one that results in the smallest objective value $x^T Rx$ *and* satisfies the constraint $c^T x > 1$, assumed in (24). In the next section, however, we show that this constraint is met for all values of the Lagrange multiplier λ greater than a minimum value, $\lambda_{\min}$. We will see that there is a single value of $\lambda > \lambda_{\min}$ that satisfies the Lagrange equations.

B. A lower bound on the Lagrange multiplier

We begin by establishing the conditions under which (9) has a solution. Assume $R = R^T \succ 0$, *i.e.*, R is symmetric and positive definite.

Lemma 1: For $A \in \mathbf{R}^{n \times n}$ full rank, there exists an $x \in \mathbf{R}^n$ for which $\|A^T x\| = c^T x - 1$ if and only if $c^T (AA^T)^{-1} c > 1$.

Proof: To prove the if direction, define

$$x(\lambda) = \left(cc^T - AA^T - \lambda^{-1} R\right)^{-1} c. \quad (32)$$

By the matrix inversion lemma, we have

$$\begin{aligned} c^T x(\lambda) - 1 &= c^T (cc^T - AA^T - \lambda^{-1} R)^{-1} c - 1 \\ &= \frac{1}{c^T (AA^T + \lambda^{-1} R)^{-1} c - 1}. \end{aligned} \quad (33)$$

For $\lambda > 0$, $c^T (AA^T + \lambda^{-1} R)^{-1} c$ is a monotonically increasing function of λ; therefore, for $c^T (AA^T)^{-1} c > 1$, there exists a $\lambda_{\min} \in \mathbf{R}^+$ for which

$$c^T (AA^T + \lambda_{\min}^{-1} R)^{-1} c = 1. \quad (34)$$

Since

$$\begin{aligned} \lim_{\lambda \to \infty} c^T x(\lambda) - 1 &= -c^T (AA^T - cc^T)^{-1} c - 1 \\ &= \frac{1}{c^T (AA^T)^{-1} c - 1} > 0, \end{aligned}$$

$c^T x(\lambda) - 1 > 0$ for all $\lambda > \lambda_{\min}$.

As in (28) and (30), let $f(\lambda) = \|A^T x\|^2 - (c^T x - 1)^2$. Examining (28), we see

$$\begin{aligned} \lim_{\lambda \to \infty} f(\lambda) &= -c^T(AA^T - cc^T)^{-1}c - 1 \\ &= \frac{1}{c^T(AA^T)^{-1}c - 1} > 0. \end{aligned}$$

Evaluating (28), we see $\lim_{\lambda \to \lambda_{\min}^+} f(\lambda) = -\infty$. For all $\lambda > \lambda_{\min}, c^T x > 1$ and $f(\lambda)$ is continuous. Hence $f(\lambda)$ assumes the value of 0, establishing the existence of a $\lambda > \lambda_{\min}$ for which $c^T x(\lambda) - 1 = \|A^T x(\lambda)\|$.

To show the only if direction, assume x satisfies $\|A^T x\| \le c^T x - 1$. This condition is equivalent to

$$z^T x \ge 1 \ \forall z \in \mathcal{E} = \{Au + c \mid \|u\| \le 1\}. \tag{35}$$

For (35) to hold, the origin cannot be contained in ellipsoid $\mathcal{E}$, which implies $c^T(AA^T)^{-1}c > 1$. □

Remark: The constraints $(c^T x - 1)^2 = \|A^T x\|^2$ and $c^T x - 1 > 0$ in (24), taken together, are equivalent to the constraint $c^T x - 1 = \|A^T x\|$ in (23). For $R = R^T \succ 0, A$ full rank and $c^T(AA^T)^{-1}c > 1$, (23) has a unique minimizer x^*. For $\lambda > \lambda_{\min}, (\lambda^{-1}R + Q)$ is full rank, and the Lagrange equation (26)

$$(\lambda^{-1}R + Q)x^* = -c$$

holds for only a single value of λ. This implies there is a unique value of $\lambda > \lambda_{\min}$, for which the secular equation (30) equals zero.

Lemma 2: For $x = -\lambda(R + \lambda Q)^{-1}c \in \mathbf{R}^n$ with $A \in \mathbf{R}^{n \times n}$ full rank, $c^T(AA^T)^{-1}c > 1$, and $\lambda > 0, c^T x > 1$ if and only if the matrix $(R + \lambda(AA^T - cc^T))$ has a negative eigenvalue.

Proof: Consider the matrix

$$M = \begin{bmatrix} \lambda^{-1}R + AA^T & c \\ c^T & 1 \end{bmatrix}.$$

We define the inertia of M as the triple $In\{M\} = \{n_+, n_-, n_0\}$, where $n+$ is the number of positive eigenvalues, n_- is the number of negative eigenvalues, and n_0 is the number of zero eigenvalues of M. See Kailath et al. [31, pp.729-730].

Since both block diagonal elements of M are invertible,

$$\begin{aligned} In\{M\} &= In\{\lambda^{-1}R + AA^T\} + In\{\Delta_1\} \\ &= In\{1\} + In\{\Delta_2\}, \end{aligned} \tag{36}$$

where $\Delta_1 = 1 - c^T(\lambda^{-1}R + AA^T)^{-1}c$, the Schur complement of the (1,1) block in M, and $\Delta_2 = \lambda^{-1}R + AA^T - cc^T$, the Schur complement of the (2,2) block in M. We conclude $c^T(\lambda^{-1}R + AA^T)^{-1}c > 1$ if and only if the matrix $(\lambda^{-1}R + AA^T - cc^T)$ has a negative eigenvalue. By the matrix inversion lemma,

$$\frac{1}{c^T(\lambda^{-1}R + AA^T)^{-1}c - 1} = -c^T(\lambda^{-1}R + AA^T - cc^T)^{-1}c - 1. \tag{37}$$

Inverting a scalar preserves its sign, therefore,

$$c^T x - 1 = -c^T(\lambda^{-1}R + AA^T - cc^T)^{-1}c - 1 > 0 \tag{38}$$

if and only if $\lambda^{-1}R + AA^T - cc^T$ has a negative eigenvalue. □

Remark: Applying Sylvester's law of inertia to equations (28) and (30), we see that

$$\lambda_{\min} = -\frac{1}{\gamma_j}, \tag{39}$$

where γ_j is the single negative generalized eigenvalue. Using this fact and (30), we can readily verify $\lim_{\lambda \to \lambda_{\min}^+} f(\lambda) = -\infty$, as stated in Lemma 1.

Two immediate consequences follow from Lemma 2. First, we may exclude from consideration any value of λ less than $\lambda_{\min}$. Second, for all $\lambda > \lambda_{\min}$, the matrix $R + \lambda Q$ has a single negative eigenvalue. We now use these facts to obtain a tighter lower bound on the value of the optimal Lagrange multiplier.

We begin by rewriting (30) as

$$\sum_{i=1}^{n} \frac{\bar{c}_i^2(-2 - \lambda\gamma_i)}{(1 + \lambda\gamma_i)^2} = \frac{1}{\lambda}. \tag{40}$$

By Sylvester's law of inertia, exactly one of the generalized eigenvalues γ in the secular equation (40) is negative. We rewrite (40) as

$$\lambda^{-1} = \frac{\bar{c}_j^2(-2 - \lambda\gamma_j)}{(1 + \lambda\gamma_j)^2} - \sum_{i \ne j} \frac{\bar{c}_i^2(2 + \lambda\gamma_i)}{(1 + \lambda\gamma_i)^2} \tag{41}$$

where j denotes the index associated with this negative eigenvalue.

A lower bound on λ can be found by ignoring the terms involving the non-negative eigenvalues in (41) and solving

$$\lambda^{-1} = \frac{\bar{c}_i^2(-2 - \lambda\gamma_j)}{(1 + \lambda\gamma_j)^2}.$$

This yields a quadratic equation in λ

$$\lambda^2(\bar{c}_j^2\gamma_j + \gamma_j^2) + 2\lambda(\gamma_j + \bar{c}_j^2) + 1 = 0, \tag{42}$$

the roots of which are given by

$$\lambda = \frac{-1 \pm |\bar{c}_j|(\gamma_j + \bar{c}_j^2)^{-1/2}}{\gamma_j}.$$

By Lemma 2, the constraint $c^T x \ge 1$ implies $R + \lambda Q$ has a single negative eigenvalue, hence, $\lambda > -\gamma_j^{-1}$. We conclude $\lambda > \hat{\lambda}$, where

$$\hat{\lambda} = \frac{-1 - |\bar{c}_j|(\gamma_j + \bar{c}_j^2)^{-1/2}}{\gamma_j}. \tag{43}$$

For any feasible beamforming problem, *i.e.*, if $Q = AA^T - cc^T$ has a negative eigenvalue, the parenthetical quantity in (43) is always nonnegative. To see this, we note that $\bar{c}_j = v_j^T R^{-\frac{1}{2}} c$, where v_j is the eigenvector associated with the negative eigenvalue γ_j. Hence, $v_j \in \mathbf{R}^n$ can be expressed as the optimal solution of

$$\begin{array}{ll} \text{minimize} & v^T R^{-\frac{1}{2}}(AA^T - cc^T)\left(R^{-\frac{1}{2}}\right)^T v \\ \text{subject to} & \|v\| = 1 \end{array} \tag{44}$$

and $\gamma_j = v_j^T R^{-\frac{1}{2}}(AA^T - cc^T)\left(R^{-\frac{1}{2}}\right)^T v_j$, the corresponding objective value. Since

$$\bar{c}_j^2 = v_j^T R^{-\frac{1}{2}} c \left(v_j^T R^{-\frac{1}{2}} c\right)^T = v_j^T R^{-\frac{1}{2}} cc^T \left(R^{-\frac{1}{2}}\right)^T v_j, \tag{45}$$

we conclude $(\gamma_j - \bar{c}_j^2) = v_j^T R^{-\frac{1}{2}} AA^T (R^{-\frac{1}{2}})^T v_j > 0$.

C. Solution of the secular equation

The secular equation (30) can be efficiently solved using Newton's method. The derivative of this secular equation with respect to λ is given by

$$f'(\lambda) = -2 \sum_{i=1}^{n} \frac{\bar{c}_i^2}{(1+\lambda\gamma_i)^3}. \tag{46}$$

As the secular equation (30) is *not* necessarily a monotonically increasing function of λ, it is useful to examine the sign of the derivative at each iteration. The Newton-Raphson method enjoys quadratic convergence if started sufficiently close to the root λ^*. The reader is referred to Dahlquist and Björck [32, §6] for details.

D. Summary and computational complexity of the RMVB computation

We summarize the algorithm below. In parentheses are approximate costs of each of the numbered steps; the actual costs will depend on the implementation and problem size [33]. As in [27] we will consider a flop to be any single floating-point operation.

RMVB computation

given R, strictly feasible A and c.

1. *Calculate* $Q \leftarrow AA^T - cc^T$. $(2n^2)$
2. *Change coordinates.* $(2n^3)$
   a. compute Cholesky factorization $LL^T = R$.
   b. compute $L^{-1/2}$.
   c. $\tilde{Q} \leftarrow L^{-1/2}Q(L^{-1/2})^T$.
3. *Eigenvalue/eigenvector computation.* $(10n^3)$
   a. compute $V\Gamma V^T = \tilde{Q}$.
4. *Change coordinates.* $(4n^2)$
   a. $\bar{q} \leftarrow V^T R^{-1/2} c$.
5. *Secular equation solution.* $(80n)$
   a. compute initial feasible point $\hat{\lambda}$
   b. find $\lambda^* > \hat{\lambda}$ for which $f(\lambda) = 0$.
6. *Compute* $x^* \leftarrow (R + \lambda^* Q)^{-1}c$ (n^3)

The computational complexity of these steps are discussed below:

1. Forming the matrix product AA^T is expensive; fortunately, it is also often avoidable. If the parameters of the uncertainty ellipsoid are stored, the shape parameter may be stored as AA^T, hence only the subtraction of the quantity cc^T need be performed, requiring $2n^2$ flops.

2. Computing the Cholesky factor L in step 2 requires $n^3/3$ flops. The resulting matrix is triangular, hence computing its inverse requires $n^3/2$ flops. Forming the matrix $\tilde{Q}$ in step 2.c requires n^3 flops.

3. Computing the eigenvalue/eigenvector decomposition is the most expensive part of the algorithm. In practice, it takes approximately $10n^3$ flops.

5. Solution of the secular equation requires minimal effort. The solution of the secular equation converges quadratically. In practice, the starting point $\hat{\lambda}$ is close to λ^*; hence, the secular equation generally converges in 7 to 10 iterations, independent of problem size.

6. Accounting for the symmetry in R and Q, computing x^* requires n^3 flops.

In comparison, the regularized beamformer requires n^3 flops. Hence the RMVB requires approximately 12 times the computational cost of the regularized beamformer. Note that this factor is independent of problem size.

III. A NUMERICAL EXAMPLE

Consider an 8-element uniform linear array, centered at the origin, in which the spacing between the elements is half of a wavelength. If we assume that the response of each element is isotropic and ignore coupling effects, the response of the array $a : \mathbf{R} \rightarrow \mathbf{C}^8$ is given by:

$$a(\theta) = \begin{bmatrix} e^{-7\phi/2} & e^{-5\phi/2} & \dots & e^{5\phi/2} & e^{7\phi/2} \end{bmatrix}^T,$$

where $\phi = \pi \sin(\theta)$ and θ is the angle of arrival. The responses of closely spaced antenna elements often differ substantially from this model.

We will compare the performance of robust beamformer with the regularized beamformer using diagonal loading. In this example, we assume a-priori, that the nominal AOA, θ_{nom}, is 120° and that the actual array response in contained in an ellipsoid $\mathcal{E}(c, P)$, whose center and configuration matrix are computed from samples of the array response, sampled at integer values, according to

$$c = \sum_{\theta=110^\circ}^{130^\circ} a(\theta) \qquad P = \frac{1}{\alpha} \sum_{\theta=110^\circ}^{130^\circ} (a(\theta) - c)\,(a(\theta) - c)^*, \tag{47}$$

with

$$\alpha = \sup_{\theta \in [110^\circ, 130^\circ]} (a(\theta) - c)^* P^{-1} (a(\theta) - c) .$$

In this example, the array response of the desired signal is taken from $\mathcal{E}(c, P)$ and the SNR at each element is 20 dB. Two uncorrelated interfering signals, s_{int1} and s_{int2} also impinge on the array. The angles of arrival of these interfering signals, θ_{int1} and θ_{int2}, are 150° and 90°; the SNRs, 40dB and 20dB, respectively. We model the received signals as:

$$y(t) = a_{\text{d}} s_{\text{d}}(t) + a(\theta_{\text{int1}}) s_{\text{int1}}(t) + a(\theta_{\text{int2}}) s_{\text{int2}}(t) + v(t), \tag{48}$$

where a_{d} denotes the array response of the desired signal, $a(\theta_{\text{int1}})$ and $a(\theta_{\text{int2}})$, the array responses for the interfering signals, $s_{\text{d}}(t)$ is the desired signal, $s_{\text{int1}}(t)$ and $s_{\text{int2}}(t)$ the interfering signals, and $v(t)$ is a complex vector of additive white noises.

In this example, we will use the analytically computed, steady state covariance, which reflects the chosen array response, and which assumes that the signals $s_{\text{d}}(t), s_{\text{int1}}(t), s_{\text{int2}}(t)$, and $v(t)$ are all uncorrelated. Let the noise power be given by $\mathbf{E}\, vv^* = \sigma_n^2 I$, where I is an $n \times n$ identity matrix and n is the number of antennas, *viz*, 8. Similarly define the powers of the desired signal, and interfering signals to be $\mathbf{E}\, s_{\text{d}} s_{\text{d}}^* = \sigma_{\text{d}}^2$, $\mathbf{E}\, s_{\text{int1}} s_{\text{int1}}^* = \sigma_{\text{int1}}^2$, and $\mathbf{E}\, s_{\text{int2}} s_{\text{int2}}^* = \sigma_{\text{int2}}^2$. Hence,

$$\frac{\sigma_{\text{d}}^2}{\sigma_n^2} = 10^2, \qquad \frac{\sigma_{\text{int1}}^2}{\sigma_n^2} = 10^4, \qquad \frac{\sigma_{\text{int2}}^2}{\sigma_n^2} = 10^2.$$

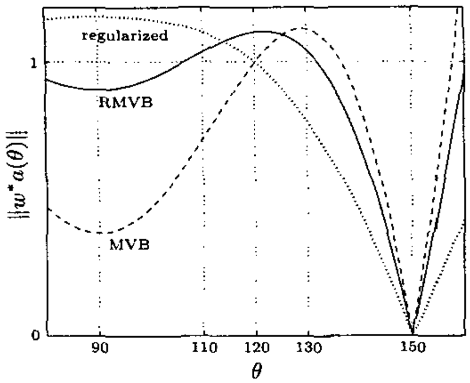

Fig. 2. The response of the MVB (Capon's method), the RMVB, and the regularized beamformer employing diagonal loading as a function of angle of arrival θ. The regularization term λ corresponds to $\frac{1}{100}$ of the maximum eigenvalue of the covariance matrix. Note that the RMVB preserves greater than unity gain for all angles of arrival in the design specification of $\theta \in [110, 130]$.

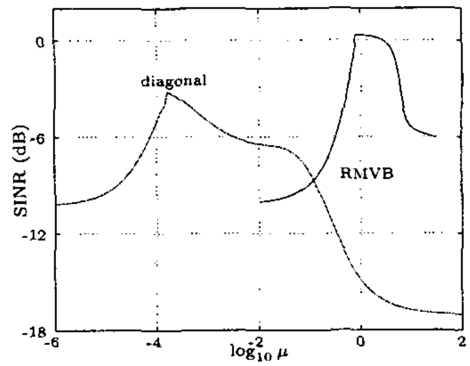

Fig. 3. The worst-case performance of the regularized beamformers based on diagonal loading (diagonal) as a function of the regularization parameter μ. The effect of scaling of the uncertainty ellipsoid used in the design of the RMVB is seen; for $\mu = 1$ the uncertainty used in designing the robust beamformer equals the actual uncertainty in the array manifold.

The expected value of the covariance matrix is given by

$$\mathbf{E}\,R = \mathbf{E}\,yy^* = \sigma_{\rm d}^2 a_{\rm d} a_{\rm d}^* + \sigma_{\rm int1}^2 a(\theta_{\rm int1})a(\theta_{\rm int1})^* + \sigma_{\rm int1}^2 a(\theta_{\rm int2})a(\theta_{\rm int2})^* + \sigma_{\rm n}^2 I. \quad (49)$$

In practice, the covariance of the received signals plus interference is often neither known nor stationary and hence must be estimated from recently received signals; as a result, the performance of beamformers are often degraded by errors in the covariance due to either small sample size or movement in the signal sources.

In Figure 2, we see the reception pattern of the array employing the MVB, the RMVB, and the regularized beamformer (10) computed using the nominal AOA and corresponding covariance matrix R. The regularization term, μ was chosen to be $\frac{1}{100}$ of the largest eigenvalue of the received covariance matrix. By design, both the MVB and the regularized beamformer have unity gain at the nominal AOA. The RMVB is seen to maintain greater than unity gain for all AOAs covered by the uncertainty ellipsoid $\mathcal{E}(c, P)$. The response of the MVB is substantially attenuated for $\theta = 110°$, when multiplicative uncertainties are considered, it is not difficult to compute scenarios in which the MVB has *zero* response to the desired signal. The response of the regularized beamformer is seen to be a detuned version of the MVB.

In Figure 3 we see the effect of changes in the regularization parameter μ on the worst-case SINRs for the regularized beamformers using diagonal loading and eigenvalue thresholding and the effect of scaling the uncertainty ellipsoid on the RMVB. Using the definition of SINR (6), we define the worst case SINR is as the minimum objective value of the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \dfrac{\sigma_{\rm d}^2\|w^*a\|^2}{w^*R_v w} \\ \text{subject to} & a \in \mathcal{E}(c, P), \end{array}$$

where $\sigma_{\rm d}^2$ denotes the power of the desired signal and R_y is the expected covariance of the interfering signals and noises:

$$R_v = \sigma_{\rm int1}^2 a(\theta_{\rm int1})a(\theta_{\rm int1})^* + \sigma_{\rm int1}^2 a(\theta_{\rm int2})a(\theta_{\rm int2})^* + \sigma_{\rm n}^2 I. \quad (50)$$

The weight vector w and covariance matrix R used in its computation reflect the chosen value of the array manifold.

For comparison, the worst-case SINR of the MVB with (three) unity mainbeam constraints at 110°, 120° and 130° is -6.24 dB. The MV-EPC beamformer was computed using the same 21 samples of the array manifold as the computation of the uncertainty ellipsoid (47); the design value for the response in each of these directions was unity. The worst-case SINRs of the rank-1 through rank-4 MV-EPC beamformers were found to be -8.67 dB, 0.23 dB, -6.21 dB, and -17.60 dB, respectively.

For diagonal loading, the parameter μ is the scale factor multiplying the identity matrix added to the covariance matrix, divided by the largest eigenvalue of the covariance matrix R. As $\mu \to 0$, the regularized beamformer reduces to Capon's method. The worst-case SINR for Capon's method is -10.26 dB. As $\mu \to \infty, w_{\rm reg} \to a(\theta_{\rm nom})$.

For the robust beamformer, we use μ to define the ratio of the size of the ellipsoid used in the beamformer computation $\mathcal{E}_{\rm design}$ divided by size of the actual array uncertainty $\mathcal{E}_{\rm actual}$. Specifically, if $\mathcal{E}_{\rm actual} = \{Au + c \mid \|u\| \leq 1\}$, $\mathcal{E}_{\rm design} = \{\mu Av + c \mid \|v\| \leq 1\}$. When the design uncertainty equals the actual, the worst-case SINR of the robust beamformer is seen to be 0.32 dB; scaling the uncertainty ellipsoid used in the design in either direction results in a decrease in the SINR of the response. We summarize the effect of differences between assumed and actual uncertainty regions on the performance of the RMVB:

- If the assumed uncertainty ellipsoid is smaller than the actual uncertainty, the minimum gain constraint will generally not be met. If the ellipsoid used in computing the

RMVB is much smaller than the actual uncertainty, the performance may degrade substantially.

• If assumed uncertainty is greater than the actual uncertainty, the performance is generally degraded, but the minimum gain in desired look direction is maintained.

The performance of the RMVB is not optimal with respect to SINR; it is optimal in the following sense. For a fixed covariance matrix R and an array response contained in an ellipsoid $\mathcal{E}$, no other vector achieves a lower weighted power out of the array while maintaining the real part of the response greater than unity for all values of the array contained in $\mathcal{E}$.

II. Conclusions

The main ideas of our approach are as follows:

• The possible values of the array manifold are covered by an ellipsoid that describes the uncertainty in the array response.

• The robust minimum variance beamformer is chosen to minimize the weighted power out of the array subject to the constraint that the gain is greater than unity for all array manifolds in the ellipsoid.

• The RMVB can be computed very efficiently using Lagrange multiplier techniques.

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