Array Signal Processing with Robust Rejection Constraints via Second-Order Cone Programming

Almir Mutapcic Seung-Jean Kim Stephen Boyd
Information Systems Laboratory
Electrical Engineering Department, Stanford University
Stanford, CA 94305-9510
{almirm,sjkim,boyd}@stanford.edu

***Abstract*— We consider the array signal processing problem of choosing the weight vector to minimize noise power, subject to a unit array gain for the desired wave, and subject to rejection constraints on interferences. We model the variations in the array response with ellipsoidal uncertainty, and take the worst-case robust optimization approach, i.e., we require the constraints to hold for all possible data in the uncertainty ellipsoid. We show that this robust array signal processing problem can be formulated as a second-order cone program, which interior-point algorithms can solve efficiently. The robust solution is demonstrated with an example.**

I. Array signal processing

We consider an array of n sensor elements. Let $a : \Omega \to \mathbf{C}^n$ be the array response to a plane wave of unit amplitude parametrized by $\theta \in \Omega$, where Ω is the set of all possible wave parameters, such as its arrival angle, wavelength, polarization, and so on. The composite output of the array is a weighted sum $w^* a(\theta)$, where $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$ is the vector of weights and $(\cdot)^*$ denotes the conjugate transpose. The magnitude $|w^* a(\theta)|$ of the array output is called the *array gain* or *array sensitivity* given the parameter θ.

We consider the following array signal processing (*i.e.*, beamforming) problem. We require a unit array gain for a wave with the desired parameter θ_1, *i.e.*, $|w^* a(\theta_1)| = 1$. We also want to impose *interference rejection* or *nulling* constraints, which are array gain constraints of the form

$$|w^* a(\theta)| \le \epsilon \qquad \forall \theta \in \Omega_{\rm rej},$$

where $\epsilon \ge 0$ is the rejection level and $\Omega_{\rm rej}$ is a set of parameters that describe waves to be rejected. We focus on the finite set, say

$$\Omega_{\rm rej} = \{\theta_2, \ldots, \theta_m\}.$$

(The infinite case can be handled approximately by discretizing the set $\Omega_{\rm rej}$.) Additionally, we want to keep the noise power $P(w) = w^* \Sigma w$ small, where Σ is the covariance of the additive noise. This objective has the statistical interpretation of minimizing the additive white Gaussian noise in the system.

The problem of choosing the weight vector that minimizes the noise power, subject to unit array gain for the desired wave and the interference rejection constraints, is given by

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & |w^* a_1| = 1 \\ & |w^* a_i| \le \epsilon, \quad i = 2, \ldots, m, \end{array} \tag{1}$$

where the variable is $w \in \mathbf{C}^n$ and the problem data are $a_i = a(\theta_i)$ for $i = 1, \ldots, m$. A solution of (1) is referred to as the *nominal optimal solution* and we denote it as $w^\star_{\rm nom}$. It is guaranteed to reject waves with parameters in $\Omega_{\rm rej}$ with a rejection level of at least ϵ.

The array processing problem (1) is not a convex optimization problem, since the equality constraint is not linear. However, we can transform it to an equivalent convex problem,

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \mathbf{Re}(w^* a_1) \ge 1 \\ & |w^* a_i| \le \epsilon, \quad i = 2, \ldots, m. \end{array} \tag{2}$$

where $\mathbf{Re}(\cdot)$ denotes the real part. This is a second-order cone problem (SOCP) when expressed in terms of the real and imaginary parts of the variables and data; it can be readily solved using the interior-point methods [11], [13]. The equivalence between problems (1) and (2) is shown in Appendix A.

II. Robust array processing with uncertain data

A. Robust array processing problem

In problem (2), we assume that the data a are perfectly known. A widely known problem is that the nominal

1-4244-0785-0/06/$20.00

optimal solution can be extremely sensitive to variations in the array response a. The goal of *robust array signal processing* or *robust beamforming* is to choose weights w such that the obtained solution performs well despite variations in a. Robust beamforming has been considered since the beginning of array signal processing [6]. One widely used robust beamforming technique is the diagonal loading method [1], [5], where an l_2-regularization term is added to the objective. More recently, ideas from (worst-case) robust optimization [2], [7], [3] have been applied to robust beamforming; for example, in robust minimum variance beamforming [14], [17], [12], in beamforming with uncertain weights [15], and in robust array pattern synthesis [18, Sec. IV]. Some other applicable robust techniques are summarized in the survey articles [10], [16].

In this paper, we consider the robust array processing with ellipsoidal uncertainty in the data a_i. We assume that for each a_i we have an ellipsoid $\mathcal{A}_i \subseteq \mathbf{C}^n$ that covers the possible values of a_i:

$$a_i \in \mathcal{A}_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\},$$

where $\bar{a}_i$ is the nominal array response, $u \in \mathbf{C}^p$, and $P_i \in \mathbf{C}^{n\times p}$ describes the shape of the ellipsoid. The norm $\|\cdot\|_2$ denotes the complex l_2-norm.

We take a worst-case robust optimization approach to problem (2) given the ellipsoidal uncertainty model described above: we require the constraints to hold for all data $a_i \in \mathcal{A}_i$. This robust optimization approach leads us to the robust array processing problem

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \mathbf{Re}(w^* a_1) \geq 1, \quad \forall a_1 \in \mathcal{A}_1 \\ & |w^* a_i| \leq \epsilon, \quad \forall a_i \in \mathcal{A}_i, \quad i = 2, \dots, m. \end{array} \tag{3}$$

A solution of (3) is referred to as the *robust optimal solution* and we denote it as $w^\star_{\text{rob}}$. This is a convex semi-infinite problem (SIP) [9] and is not tractable in this formulation.

B. SOCP formulation

The main contribution of this paper is to show that the SIP (3) can be reformulated as

$$\begin{array}{ll} \text{minimize} & P(w) \\ \text{subject to} & \mathbf{Re}(w^* \bar{a}_1) \geq 1 + \|P_1^* w\|_2 \\ & |w^* \bar{a}_i| + \|P_i^* w\|_2 \leq \epsilon, \quad i = 2, \dots, m. \end{array} \tag{4}$$

This problem becomes an SOCP when expressed in terms of the real and imaginary parts of the variables and data, and so can be efficiently solved using interior-point methods [13].

The equivalence between (3) and (4) follows directly from the following two observations. Consider an ellipsoid $\mathcal{A} = \{\bar{a} + Pu \mid \|u\|_2 \leq 1\}$, where $\bar{a} \in \mathbf{C}^n$, $P \in \mathbf{C}^{n\times p}$, and $u \in \mathbf{C}^p$. Then,

- $\mathbf{Re}(w^* a) \geq 1$ for all $a \in \mathcal{A}$ if and only if
$$\mathbf{Re}(w^* \bar{a}) \geq 1 + \|P^* w\|_2. \tag{5}$$
- $|w^* a| \leq \epsilon$ for all $a \in \mathcal{A}$ if
$$|w^* \bar{a}| + \|P^* w\|_2 \leq \epsilon. \tag{6}$$

The first result is widely used in robust beamforming, and follows from the Cauchy-Schwarz inequality; see, *e.g.*, [14]. The second result is also a consequence of the Cauchy-Schwarz inequality, as shown below.

Observe that, for any $a \in \mathcal{A}$, we have a chain of inequalities

$$\begin{aligned} |w^* a| &= |w^*(\bar{a} + Pu)| \\ &\leq |w^* \bar{a}| + |(P^* w)^* u| \\ &\leq |w^* \bar{a}| + \|P^* w\|_2. \end{aligned}$$

The first inequality comes from the triangle inequality, while the second one comes from the Cauchy-Schwarz. Moreover, equality holds here with the choice

$$u = \frac{P^* w}{\|P^* w\|_2} e^{i\phi}, \qquad \phi = \angle (w^* \bar{a}).$$

Therefore, we have

$$\sup_{a\in\mathcal{A}} |w^* a| = |w^* \bar{a}| + \|P^* w\|_2.$$

This completes the proof since $|w^* a| \leq \epsilon$ for all $a \in \mathcal{A}$ if and only if $\sup_{a\in\mathcal{A}} |w^* a| \leq \epsilon$.

III. Example

We consider an array with $n = 20$ sensor elements in a plane. The spacing between the sensors is about 0.5λ, where λ is the wavelength of plane waves arriving from angles $\theta \in \Omega = [0°, 360°]$. We use a simple model for the array response,

$$a(\theta)_j = \exp(2\pi i/\lambda(x_j \cos\theta + y_j \sin\theta)),$$

where (x_j, y_j) is the location of the jth sensor element. We take the desired direction $\theta_1 = 45°$ and we consider a finite rejection set $\Omega_{\text{rej}} = \{10°, 65°, 120°, 200°\}$ with the rejection level $\epsilon = 0.05$ (-26 dB). Figure 1 shows our sensor array and the wave angles.

We conduct the following numerical experiment. We consider the objective function $P(w) = w^* w$, *i.e.*, we minimize the effect of the additive white Gaussian noise in the system. We keep the constraint for unit array gain in the desired direction certain, *i.e.*, $\mathcal{A}_1 = \{\bar{a}_1\}$,

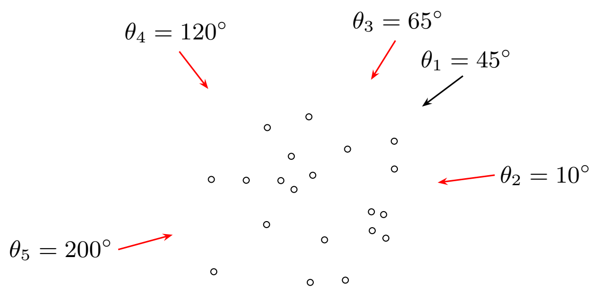

Fig. 1. A sensor array and directions of impinging plane waves.

and introduce uncertainty in the interference rejection constraints. We consider isometric uncertainties centered around the nominal array responses, *i.e.*, the uncertainty matrices $P_2, \ldots, P_5$ are scaled identities $\rho I \in \mathbf{C}^{20\times 20}$, where $\rho > 0$ is the *measure* of the uncertainty.

We solve the nominal array processing problem (2) and a family of robust array processing problems (4) for various values of ρ using CVX [8]. For a particular value of ρ, we compute the worst-case interference rejection level $R_{\rm wc}$, defined as

$$R_{\rm wc}(w) = \max_i \sup_{a_i \in \mathcal{A}_i} |w^* a_i| = \max_i |w^* \bar{a}_i| + \rho \|w\|_2, \quad i = 2, \ldots, 5 \tag{7}$$

for both the nominal optimal $w^\star_{\rm nom}$ and the robust optimal solutions $w^\star_{\rm rob}$. The worst-case rejection level versus ρ is shown in figure 2. We note that the robust solution always rejects the interference waves with rejection level of at least $\epsilon = 0.05$ (-26 dB), while the nominal optimal solution violates these constraints in the worst-case scenario by an amount of $\rho \|w^\star_{\rm nom}\|_2$ as predicted by (7). Figure 3 shows increase in the optimal value for the robust problem (4) with respect to the nominal problem (2). This increase in the objective value is the price we have to pay in order to guarantee robust rejection of the interference waves. For example, when $\rho = 0.15$, we have a 2% increase in the objective value, while the nominal optimal solution violates a rejection constraint by 0.036.

IV. Conclusions

In this paper, we have shown how one can take into account the variations in the array responses to the desired wave and interferences, using the worst-case robust optimization approach with an ellipsoidal uncertainty model of data. We have restricted ourselves to an ellipsoidal uncertainty model. The convex formulation can be easily extended to general complex l_p-norm ball uncertainty description (with $p \geq 1$), and one can still compute the worst-case gain analytically. The resulting robust optimization problem is still a convex problem which interior-point algorithms can solve with great efficiency.

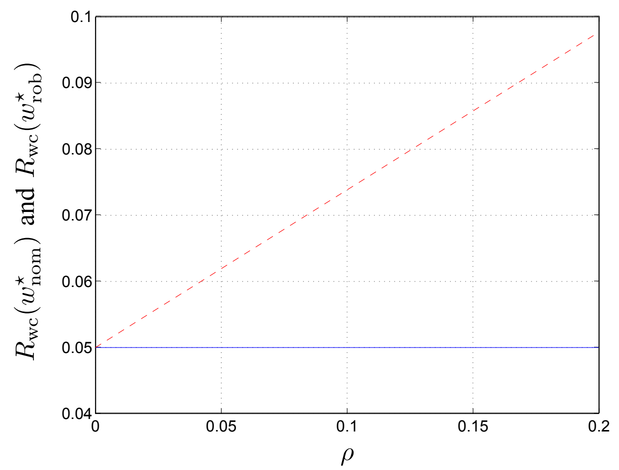

Fig. 2. Worst-case interference rejection levels for the nominal optimal solution $w^\star_{\rm nom}$ (red dashed curve) and the robust optimal solution $w^\star_{\rm rob}$ (blue solid curve).

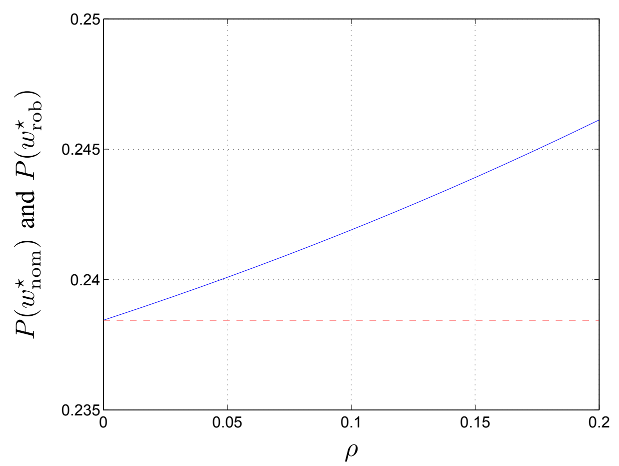

Fig. 3. Optimal value of the objective function $P(w) = w^* w$ for the nominal optimal solution $w^\star_{\rm nom}$ (red dashed curve) and the robust optimal solution $w^\star_{\rm rob}$ (blue solid curve).

Acknowledgments

This material is supported in part by Dr. Dennis Healy, DARPA/MTO, under ONR grant #N00014-05-1-0700, by the National Science Foundation under grant #0423905, by the Air Force Office of Scientific Research under grant #F49620-01-1-0365, and by MARCO Focus center for Circuit & System Solutions contract #2003-CT-888. The authors thank Arogyaswami Paulraj for helpful comments and suggestions.

APPENDIX A
PROBLEM EQUIVALENCE

We establish the equivalence of the problems (1) and (2). For any $\alpha \in \mathbf{C}$ and $i = 1, \ldots, m$, we have

$$P(\alpha w) = |\alpha| P(w), \quad |(\alpha w)^* a_i| = |\alpha|\, |w^* a_i|\,.$$

Suppose w is feasible for (2). Since $\mathbf{Re}(w^* a_1) \geq 1$, we have $|w^* a_1| \geq 1$, so $\tilde{w} = (1/|w^* a_1|)w$ satisfies $|\tilde{w}^* a_1| = 1$ and $|\tilde{w}^* a_i| \leq \epsilon$ for $i = 2, \ldots, m$. Thus, $\tilde{w}$ is feasible for (1), and furthermore, it satisfies

$$P(\tilde{w}) = \frac{P(w)}{|w^* a_1|} \leq P(w).$$

Conversely, suppose that $\tilde{w}$ is feasible for (1), *i.e.*, $|\tilde{w}^* a_1| = 1$ and $|\tilde{w}^* a_i| \leq \epsilon$ for $i = 2, \ldots, m$. Then the point

$$w = \frac{\tilde{w}^* a_1}{|\tilde{w}^* a_1|} \tilde{w}$$

is feasible for (2), and satisfies $P(w) = P(\tilde{w})$. Thus from any feasible point for either problem, we can construct a feasible point for the other problem, with equal or lower objective value, so we conclude they are equivalent [4, Chap. 4].

REFERENCES

[1] Y. Abramovich. Controlled method for adaptive optimization of filters using the criterion of maximum SNR. *Radio Engineering and Electronic Physics*, 26:87–95, March 1981.

[2] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[3] A. Ben-Tal and A. Nemirovski. Robust optimization: methodology and applications. *Mathematical Programming Series B*, 92(3):453–480, May 2002.

[4] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[5] B. Carlson. Covariance matrix estimation errors and diagonal loading in adaptive arrays. *IEEE Transactions on Aerospace Electronics and Systems*, 24(7):397–401, July 1988.

[6] H. Cox, R. Zeskind, and M. Owen. Robust adaptive beamforming. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 35:1365–1376, October 1987.

[7] L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal on Matrix Analysis and Applications*, 18(4):1035–1064, 1997.

[8] M. Grant, S. Boyd, and Y. Ye. CVX: Matlab software for disciplined convex programming, version 1.0 beta 3. Available at `www.stanford.edu/~boyd/cvx/`, June 2006.

[9] R. Hettich and K. Kortanek. Semi-infinite programming: theory, methods, and applications. *SIAM Review*, 1993.

[10] S. Kassam and V. Poor. Robust techniques for signal processing: A survey. *Proceedings of IEEE*, 73:433–481, 1985.

[11] H. Lebret and S. Boyd. Antenna array pattern synthesis via convex optimization. *IEEE Transactions on Signal Processing*, 45(3):526–532, June 1997.

[12] J. Li, P. Stoica, and Z. Wang. On robust Capon beamforming and diagonal loading. *IEEE Transactions on Signal Processing*, 51(7):1702–1715, July 2003.

[13] M. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret. Applications of second-order cone programming. *Linear Algebra and Its Applications*, 284:193–228, 1998.

[14] R. Lorenz and S. Boyd. Robust minimum variance beamforming. *IEEE Transactions on Signal Processing*, 53(5):1684–1696, 2005.

[15] A. Mutapcic, S.-J. Kim, and S. Boyd. Beamforming with uncertain weights, 2006. To appear in *IEEE Signal Processing Letters*.

[16] S. Verdú and H. Poor. On minimax robustness: A general approach and applications. *IEEE Transactions on Information Theory*, 30(2):328–340, 1984.

[17] S. Vorobyov, A. Gershman, and Z.-Q. Luo. Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem. *IEEE Transactions on Signal Processing*, 51(4):313–324, April 2003.

[18] F. Wang, V. Balakrishnan, P. Zhou, J. Chen, R. Yang, and C. Frank. Optimal array pattern synthesis using semidefinite programming. *IEEE Transactions on Signal Processing*, 51:1172–1183, 2003.