Two-Fund Separation under Model Mis-Specification

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Abstract

The two-fund separation theorem tells us that an investor with quadratic utility can separate her asset allocation decision into two steps: First, find the tangency portfolio (TP), *i.e.*, the portfolio of risky assets that maximizes the Sharpe ratio (SR); and then, decide on the mix of the TP and the risk-free asset, depending on the investor's attitude toward risk. In this paper, we describe an extension of the two-fund separation theorem that takes into account uncertainty in the model parameters (*i.e.*, the expected return vector and covariance of asset returns) and uncertainty aversion of investors. The extension tells us that when the uncertainty model is convex, an investor with quadratic utility and uncertainty aversion can separate her investment problem into two steps: First, find the portfolio of risky assets that maximizes the worst-case SR (over all possible asset return statistics); and then, decide on the mix of this risky portfolio and the risk-free asset, depending on the investor's attitude toward risk. The risky portfolio is the TP corresponding to the least favorable asset return statistics, with portfolio weights chosen optimally. We will show that the least favorable statistics (and the associated TP) can be found efficiently by solving a convex optimization problem.

1 Introduction

The two-fund separation theorem [53] is a central result in modern portfolio theory pioneered by Markowitz [39, 40]. It tells us that the risk-return pair of any admissible or feasible portfolio cannot lie above the capital market line (CML) in the risk-return space, obtained by combining the risk-free asset and the portfolio that maximizes the Sharpe ration (SR). An important implication is that an investor can separate her asset allocation decision into two steps: First, find the portfolio of risky assets that maximizes the SR; then, decide on the mix of the optimal risky portfolio and the risk-free asset, depending on her attitude toward risk. Sharpe [50] and Lintner [34] derive the implications of the two-fund separation property for equilibrium prices, which is known as the capital asset pricing model (CAPM).

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The two-step asset allocation process is based on the assumption that there is no model uncertainty or model mis-specification, *i.e.*, the input data or parameters (the mean vector and covariance matrix of asset returns) are perfectly known. These input parameters are typically empirically estimated from historical data of asset returns, or from extensive analysis of various types of information about the assets and macro-economic conditions. Due to inevitable imperfections in the analysis and estimation procedure, the parameters are estimated with error. The standard two-step asset allocation process can be very sensitive to the estimation error: Portfolios constructed on the basis of estimated values of the parameters cam have very poor performance for another set of parameters that is similar and statistically indistinguishable from the one used in the allocation decision [28]. The literature on the sensitivity problem of MV allocation to estimation error or model uncertainty is huge; see, [3, 4, 8, 7, 23, 27, 43] to name a few.

A variety of approaches have been suggested for alleviating the sensitivity problem in MV asset allocation. The list includes imposing constraints such as no short-sales constraints [26], the resampling approach [43], Bayesian approaches [1, 9, 31, 44, 45, 56], the shrinkage approach [13, 55, 24], the empirical Bayes approach [19], the Black-Litterman approach [4] (which allows investors to incorporate economic views into the asset allocation process), and the worst-case approach [11, 16, 20, 25, 29, 28, 54, 47, 49]. The reader is referred to the expository article [6] for an overview of these approaches.

This paper contributes to the literature on the worst-case approach. This approach is related to the view of Knight [32] that we should distinguish between "uncertainty" (ambiguous probabilities) and "risk" (precisely known probabilities), and model uncertainty, or more precisely, the investors' assessment of model uncertainty, which cannot be represented by a probability prior. Its axiomatic foundation is laid out in [21, 15] which formally describe the max-min expected utility framework, in which an investor with ambiguity or uncertainty aversion would compute the expected utility by using the "worst" parameter set over the set of all possible parameters and chooses its strategy to maximize the worst-case expected utility. More generally, the worst-case approach explicitly incorporates a model of data uncertainty in the formulation of a portfolio selection problem, and optimizes for the worst-case scenario under this model; see, *e.g.*, [11, 16, 14, 22, 20, 25, 29, 28, 35, 54, 47, 49]. The reader is referred to a recent survey [18] and monographs [17, 42, 48] on robust asset allocation.

In this paper, we describe an extension of the asset allocation process, that takes into account model mis-specification and investor uncertainty aversion. We show that when the uncertainty model is convex, and the investor's utility is quadratic, she separate her investment problem into two steps: Find the portfolio of risky assets that maximizes the worst-case SR (over all possible asset return statistics); then, decide on the mix of the risky portfolio and the risk-free asset, considering her risk aversion. The risky portfolio is the TP of the least favorable asset return statistics, with the portfolio weights chosen optimally. We will also show that the least favorable statistics (and the associated TP) can be found efficiently by solving a convex optimization problem.

We give a review of the two-fund separation theorem in Section 2, to set up our notation and compare it to the extension we describe in Section 3. We illustrate the extension with a numerical example in Section 4. We give our conclusions in Section 5.

2 Two-fund separation in MV asset allocation

We have n risky assets, denoted $1, \ldots, n$, and a risk-free asset, denoted $n+1$. These assets are held over a period of time. We use a_i to denote the relative price change of asset i over the period, that is, its change in price over the period divided by its price at the beginning of the period. Let $\mu = \mathbf{E}a$ and $\Sigma = \mathbf{E}(a-\mu)(a-\mu)^T$ denote the mean and covariance of $a = (a_1, \ldots, a_n)$, where $\mathbf{E}$ denotes the expectation operation, and let $\mu_{\rm rf}$ denote the return of the risk-free asset $n+1$. We assume that Σ is positive definite.

A portfolio will be denoted as a vector $x \in \mathbf{R}^{n+1}$, with x_i denoting the amount invested in asset i, with a long position in asset i corresponding to $x_i > 0$, and a short position in asset i corresponding to $x_i < 0$. (For the risk-free asset, $x_{n+1} < 0$ corresponds to borrowing at the interest rate $\mu_{\rm rf}$.) We assume the portfolio satisfies the budget constraint $\mathbf{1}^T x = 1$, where $\mathbf{1}$ denotes the vector of all ones.

The portfolio x can be represented as an affine combination of $(w, 0)$, a portfolio consisting only of risky assets, and the portfolio $(0, 1)$, consisting only of the risk-free asset:

$$x = ((1-\theta)w, \theta) = (1-\theta)(w, 0) + \theta(0, 1) \in \mathbf{R}^{n+1}.$$

(For column vectors u and v, (u, v) is the column vector obtained by stacking u on top of v.) Evidently, we have $\theta = x_{n+1}$, and $w = (x_1, \ldots, x_n)/(1-\theta)$, for $\theta \neq 1$, and $w = 0$, for $\theta = 1$. The all risky asset portfolio $w \in \mathbf{R}^n$ satisfies the portfolio budget constraint $\mathbf{1}^T w = 1$, and θ can be interpreted as the fraction of the risk-free asset, and $1-\theta$ as the leverage of the risky portfolio w. When $\theta < 0$, the investor leverages the risky portfolio by borrowing at the risk-free rate.

Let $\mathcal{W} \subset \mathbf{R}^n$ denote the set of all admissible or feasible portfolios w that consist of the risky assets $a_1, \ldots, a_n$ and satisfy the budge constraint $\mathbf{1}^T w = 1$. We assume that the set $\mathcal{W}$ is convex. The set $\mathcal{W}$ can represent a wide variety of asset allocation constraints including portfolio diversification and short-selling constraints [35, 36]. The set of all admissible or feasible portfolios of the assets $a_1, \ldots, a_{n+1}$ is

$$\mathcal{X} = \{((1-\theta)w, \theta) \in \mathbf{R}^{n+1} \mid w \in \mathcal{W},\ \theta \leq 1\},$$

where the constraint $\theta \leq 1$ is imposed to rule out a short selling position in the risky portfolio w. This set is convex; see Appendix A.1 for the proof.

2.1 Risk and return

At the end of the period, the return of a portfolio $x = ((1-\theta)w, \theta)$ is a (scalar) random variable $(1-\theta)\sum_{i=1}^n w_i a_i + \theta a_{n+1}$. The mean return is

$$r(x, \mu) = (1-\theta)w^T \mu + \theta \mu_{\rm rf},$$

and the return volatility or risk, measured by the standard deviation, is

$$\sigma(x, \Sigma) = |1-\theta|(w^T \Sigma w)^{1/2} = (1-\theta)(w^T \Sigma w)^{1/2}$$

since we assume $\theta \leq 1$. For a portfolio of the form $x = (w, 0)$, we use the shorthand notation

$$r(w, \mu) = r(w, 0, \mu), \qquad \sigma(w, \Sigma) = \sigma(w, 0, \Sigma).$$

As the leverage of the risky portfolio w is changed, the risk and return of the portfolio of $x = ((1-\theta)x, \theta)$ vary as

$$r((1-\theta)w, \theta, \mu) = (1-\theta)r(w, \mu) + \theta\mu_{\mathrm{rf}}, \qquad \sigma((1-\theta)w, \theta, \Sigma) = (1-\theta)\sigma(w, \Sigma),$$

which traces a line, parametrized by θ, in risk-return space.

The choice of a portfolio involves a trade-off between risk and return [39]. The optimal trade-off achieved by admissible portfolios of risky assets $a_1, \ldots, a_n$ is described by the curve

$$f_{\mu,\Sigma}(\sigma) = \sup_{w \in \mathcal{W},\ (w^T \Sigma w)^{1/2} \leq \sigma} w^T \mu, \tag{1}$$

which is called the (MV or Markowitz) efficient frontier (EF) for the risk assets. Each point on the EF corresponds to the risk and return of the portfolio that maximizes the mean return subject to achieving a maximum acceptable volatility level σ and satisfying the asset allocation and portfolio budget constraints. A basic property of the EF is that it is increasing and concave.

When the risk-free asset is included, the optimal trade-off analysis becomes simpler. It suffices to find a single fund (portfolio) of risky assets; any MV efficient portfolio can then be constructed as a combination of the fund and the risk-free asset, as first observed by Tobin [53]. In this case, the EF is a straight line.

2.2 SR maximization and optimal capital allocation line

The reward-to-variability or Sharpe ratio [51, 52] of a portfolio $x = ((1-\theta)w, \theta)$, which is denoted as $S(x, \mu, \Sigma)$, is its excess return (relative to the risk free rate) divided by the standard deviation of its excess return:

$$S(x, \mu, \Sigma) = \frac{r(x, \mu) - \mu_{\mathrm{rf}}}{\sigma(x, \Sigma)}.$$

For a portfolio $(w, 0)$ of risky assets only, we use the shorthand notation

$$S(w, \mu, \Sigma) = S(w, 0, \mu, \Sigma).$$

The SR of x is invariant to the leverage of the risky portfolio w: for $\theta < 1$,

$$S((1-\theta)w, \theta, \mu, \Sigma) = S(w, \mu, \Sigma) = \frac{w^T \mu - \mu_{\mathrm{rf}}}{\sqrt{w^T \Sigma w}}.$$

The problem of finding the portfolio of risky assets that maximizes the SR can be cast as

$$\begin{array}{ll} \text{maximize} & S(w,\mu,\Sigma) \\ \text{subject to} & w \in \mathcal{W}, \end{array} \tag{2}$$

where the variable is $w \in \mathbf{R}^n$ and the problem data are μ and Σ. This problem is called the SR maximization problem (SRMP). With general convex asset allocation constraints, it can be reformulated as a convex optimization problem [22, 30, 54]. Its optimal value is called the market price of risk. We use $S_{\rm mp}(\mu,\Sigma)$ to denote the optimal value, as a function of the parameters μ and Σ:

$$S_{\rm mp}(\mu,\Sigma) = \sup_{w \in \mathcal{W}} S(w,\mu,\Sigma).$$

As the fraction θ of the risk-free asset decreases from 1, the risk σ and the return r of $x = ((1-\theta)w, \theta)$ move along the line

$$r = \mu_{\rm rf} + S(w,\mu,\Sigma)\sigma$$

in the (σ, r) space, which is called the capital allocation line (CAL) of w. The line

$$r = \mu_{\rm rf} + S_{\rm mp}(\mu,\Sigma)\sigma \tag{3}$$

is called the optimal CAL or capital market line (CML). When the SRMP has a solution $w^\star$, the optimal CAL is tangential to the efficient frontier at the point $(\sigma_{\rm tan}, r_{\rm tan})$ where $\sigma_{\rm tan}$ and $r_{\rm tan}$ are the risk and return of the portfolio $w^\star$. For this reason, the portfolio $w^\star$ is called the tangency portfolio. Otherwise, the efficient frontier has an (upper) asymptote and the optimal CAL is parallel to the asymptote. Figure 1 illustrates this key result in modern portfolio theory.

2.3 Optimal allocation and two-fund separation

The following proposition follows from the observations made above.

Proposition 1 (Two-fund separation [53])**.** *The CML is the optimal trade-off curve between risk and return for portfolios $x \in \mathcal{X}$:*

- *The risk σ and the return r of any admissible portfolio $x \in \mathcal{X}$ cannot lie above the optimal CAL:*

$$r \le \mu_{\rm rf} + S_{\rm mp}(\mu,\Sigma)\sigma. \tag{4}$$

- *If the SR is maximized by $w^\star \in \mathcal{W}$, then for any $\theta \le 1$, the risk σ and the return r of $x = ((1-\theta)w^\star, \theta)$ lie on the optimal CAL:*

$$r = \mu_{\rm rf} + S_{\rm mp}(\mu,\Sigma)\sigma. \tag{5}$$

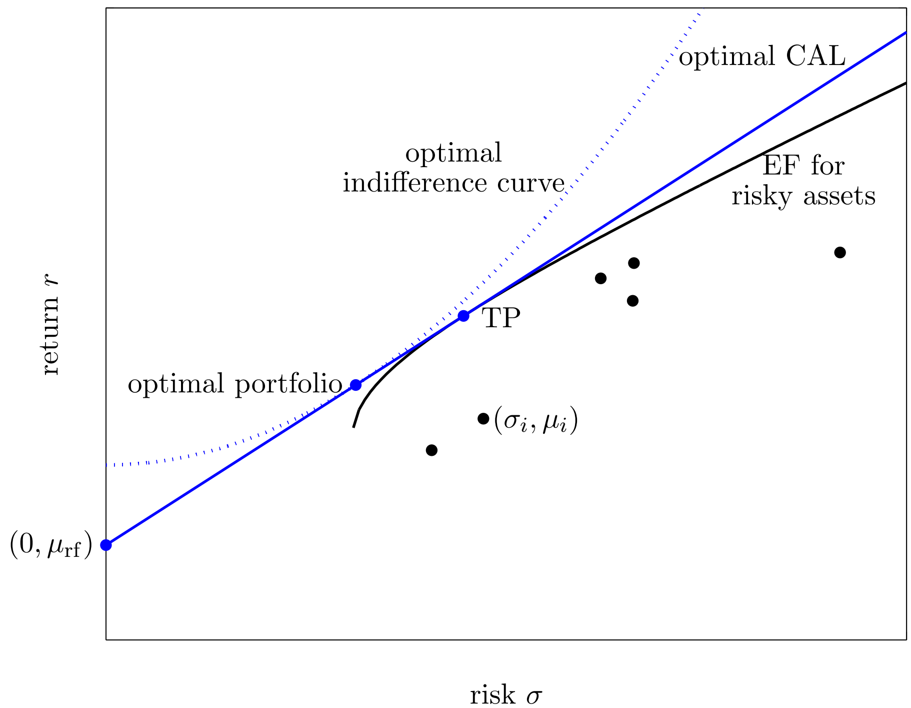

Figure 1: Two-fund separation and expected quadratic utility maximization.

This proposition tells us that when lending or borrowing at the risk-free rate is allowed, the best risk-return trade-off can be achieved by combining the two funds (portfolios): $(w^\star, 0) \in \mathbf{R}^{n+1}$, consisting only of the risky assets, and $(0, 1) \in \mathbf{R}^{n+1}$, consisting only of the risk-free asset. For this reason, Proposition 1 is called the two-fund separation theorem, or the one-fund theorem [37], since it means we need only a single fund of risky assets, to recreate any point on efficient frontier by combining it with the risk-free asset.

To decide on the mix of the risky portfolio and the risk-free asset, we take into account the attitude of the investor toward risk. We consider an investor whose utility can be modeled as an expected quadratic utility function

$$U(x, \mu, \Sigma) = \mathbf{E}(x^T a) - \frac{\gamma}{2}\mathbf{V}(x^T a) = x^T \left[\begin{array}{c} \mu \\ \mu_{\rm rf} \end{array} \right] - \frac{\gamma}{2} x^T \left[\begin{array}{cc} \Sigma & 0 \\ 0 & 0 \end{array} \right] x, \qquad (6)$$

where $\gamma > 0$ is a positive constant related to the investor's attitude toward risk and $\mathbf{V}(x^T a)$ is the variance of the random variable $x^T a$. For such an investor, the portfolio that maximizes expected utility can be found by solving the problem

$$\begin{array}{ll} \mbox{minimize} & U(x, \mu, \Sigma) \\ \mbox{subject to} & x \in \mathcal{X}. \end{array} \qquad (7)$$

This expected utility maximization problem (EQUMP) is a convex problem that can be solved efficiently [5]. (In particular, when the constraint set $\mathcal{X}$ is polyhedral, this problem is a convex quadratic program (QP).)

The two-fund separation property allows us to find a closed-form solution for the EQUMP.

Proposition 2. *The EQUMP (7) has a solution if and only if the SRMP (2) has a solution. If $w^\star \in \mathcal{W}$ solves the SRMP (2), then the portfolio $x^\star = ((1-\theta^\star)w^\star, \theta^\star) \in \mathbf{R}^{n+1}$, with*

$$\theta^\star = 1 - \frac{1}{\gamma}\frac{w^{\star T}\mu - \mu_{\rm rf}}{w^{\star T}\Sigma w^\star}, \tag{8}$$

is the unique solution to (7).

Figure 1 illustrates the basic results in modern portfolio theory given above. The dotted curve is the optimal optimal indifference curve

$$r = U^\star + (\gamma/2)\sigma^2, \qquad U^\star = \sup_{x\in\mathcal{X}} U(x,\mu,\Sigma),$$

consisting of risk-return pairs which achieve the highest level of utility attainable subject to the asset allocation and budget constraints. The point at which the optimal indifference curve is tangential to the CML corresponds to the risk and return $(\sigma(x^\star,\Sigma), r(x^\star,\mu))$ of the portfolio $x^\star = ((1-\theta^\star)w^\star, \theta^\star)$ that maximizes the expected quadratic utility.

3 Two-fund separation under model mis-specification

We now consider the case when the input parameters in the asset allocation model are not known, *i.e.*, we take into account model mis-specification.

3.1 Risk and return under model mis-specification

We use $\mathcal{U} \subseteq \mathbf{R}^n \times \mathbf{S}^n_{++}$ to denote the set of possible input parameters. This set could represent, for example, the set of parameter values that are hard to distinguish from the baseline or nominal values, based on historical returns. Here $\mathbf{S}^n_{++}$ denotes the set of all $n\times n$ symmetric positive definite matrices; $\mathbf{S}^n$ denotes the set of all $n \times n$ symmetric matrices.

With model uncertainty, the risk and return profile of a portfolio x is described by a set in the risk-return plane. We use $\mathcal{P}(x)$ to denote the set of possible risk-return pairs of a portfolio $x = ((1-\theta)w, \theta)$, consistent with the uncertainty model $\mathcal{U}$:

$$\mathcal{P}(x) = \{(r(x,\mu), \sigma(x,\Sigma)) \mid (\mu,\Sigma) \in \mathcal{U}\}.$$

As the leverage of the risky portfolio w is changed, the set $\mathcal{P}((1-\theta)w,\theta)$ varies as

$$\mathcal{P}((1-\theta)w,\theta) = (1-\theta)\mathcal{P}(w) + \theta(0,\mu_{\rm rf}), \tag{9}$$

where we use the shorthand notation $\mathcal{P}(w) = \mathcal{P}(w,0)$, $\mathcal{A} + (u,v)$ means the translation of the set $\mathcal{A}$ by the vector (u,v), and αA means the scaling of $\mathcal{A}$ by α. As the risky portfolio is more leveraged, the risk and return set of $x = ((1-\theta)w,\theta)$ moves along a line, and grows proportionally. Figure 2 illustrates the dispersion effect due to the leverage.

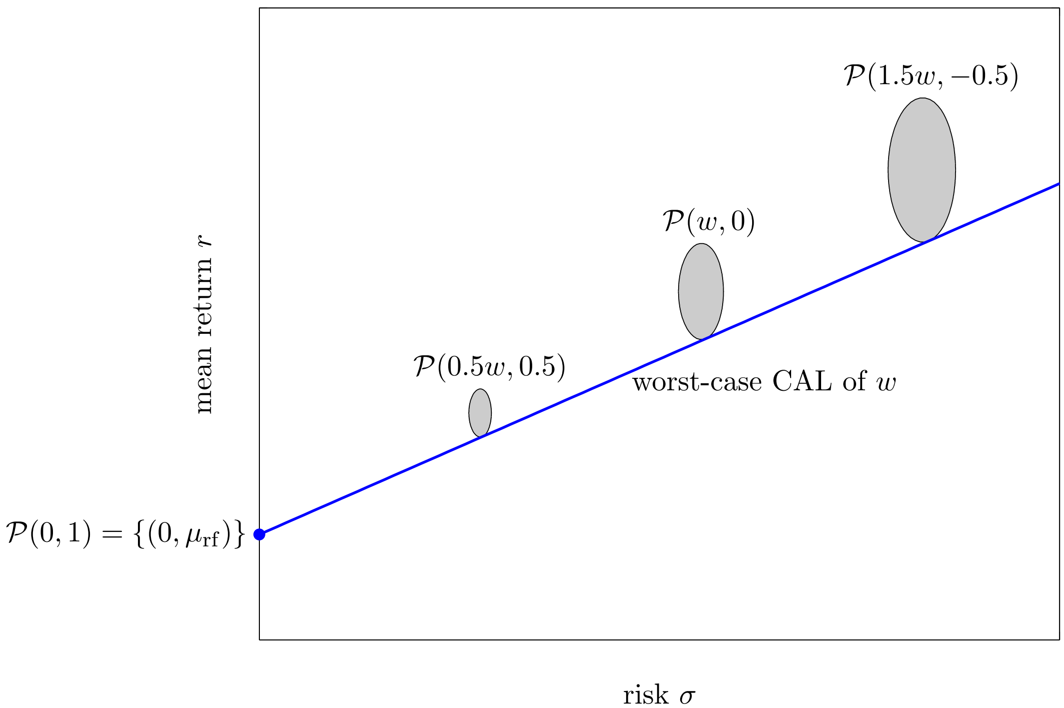

Figure 2: Risk-return sets of portfolios with different leverages of a portfolio w of risky assets under model uncertainty.

3.2 Worst-case SR analysis and optimization

We introduce the counterparts of several definitions, such as the SR and CAL, in MV analysis under model mis-specification. We will then give a review of the minimax result for the SR proved in [30], along with a geometric interpretation.

We make the following assumption:

$$\inf_{(\mu,\Sigma)\in\mathcal{U}} \bar{w}^T\mu > \mu_{\mathrm{rf}}. \tag{10}$$

This assumption means that there exists an admissible portfolio $\bar{w} \in \mathcal{W}$ of risky assets whose worst-case mean excess return is positive.

Worst-case SR analysis

For a given portfolio x, the worst-case SR analysis problem can be formulated as

$$\begin{array}{ll} \text{minimize} & S(x,\mu,\Sigma) \\ \text{subject to} & (\mu,\Sigma) \in \mathcal{U}, \end{array} \tag{11}$$

in which μ and Σ are the variables and x is fixed. Here, we compute the 'worst' (smallest) SR of the given portfolio when the mean return vector and covariance vary over the set $\mathcal{U}$. The optimal value of this problem is called the *worst-case Sharpe ratio* and is denoted $S_{\rm wc}(x)$. The worst-case SR of a portfolio x is equal to the minimum slope of the lines which connect $(0, \mu_{\rm rf})$ and points in the set $\mathcal{P}(x)$:

$$S_{\rm wc}(x) = \inf_{(\sigma,r)\in\mathcal{P}(x)} \frac{r-\mu_{\rm rf}}{\sigma}.$$

Optimal μ and Σ for the problem (11) are called a *worst-case mean return* and a *worst-case covariance* of x, respectively.

The worst-case SR satisfies

$$S_{\rm wc}((1-\theta)x, \theta) = S_{\rm wc}(w), \tag{12}$$

for $\theta < 1$, where we use the shorthand notation $S_{\rm wc}(w)$ for $S_{\rm wc}(w, 0)$. Thus, the worst-case SR is invariant with respect to the leverage of the risky portfolio.

The line

$$r = \mu_{\rm rf} + S_{\rm wc}(w)\sigma$$

has the smallest slope among all CALs computed with model parameters in the set $\mathcal{U}$. The line is called the worst-case CAL of w. For any $\theta \leq 1$, the risk-return set $\mathcal{P}(x)$ of $x = ((1-\theta)w, \theta)$ lies on or above the line and the set and the line meet at the point $(\sigma(x, \Sigma_{\rm wc}), r(x, \mu_{\rm wc}))$.

Figure 2 illustrates the definitions introduced above.

A zero-sum game involving the SR

We consider the continuous zero-sum game in which the investor attempts to choose w from the convex set $\mathcal{W}$, to maximize the SR, and her opponent attempts to choose (μ, Σ) from the convex set $\mathcal{U}$, to minimize it. The game is associated with the following two problems:

- *Worst-case SR maximization problem.* Find an admissible portfolio w that maximizes the worst-case SR:

$$\begin{array}{ll} \text{maximize} & \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w, \mu, \Sigma) \\ \text{subject to} & w \in \mathcal{W}. \end{array} \tag{13}$$

- *Worst-case market price of risk analysis problem (MPRAP).* Find the least favorable asset return statistics, over the uncertainty set $\mathcal{U}$, with optimal portfolio weights:

$$\begin{array}{ll} \text{minimize} & \sup_{w\in\mathcal{W}} S(w, \mu, \Sigma) \\ \text{subject to} & (\mu, \Sigma) \in \mathcal{U}. \end{array} \tag{14}$$

We first address the questions of existence and uniqueness in these two problems. The worst-case MPRAP (14) always has a solution, which need not be unique. The worst-case SRMP (13) need not have a solution; but when it has a solution, it is unique. The proofs are in Appendix A.2.

These two problems lead us to define two robust counterparts of the optimal CAL (3). In the (σ, r) space, the line

$$r = \mu_{\rm rf} + \sup_{w \in \mathcal{W}} \inf_{(\mu,\Sigma) \in \mathcal{U}} S(w, \mu, \Sigma)\sigma \tag{15}$$

is called the robust optimal CAL. The line

$$r = \mu_{\rm rf} + \inf_{(\mu,\Sigma) \in \mathcal{U}} \sup_{w \in \mathcal{W}} S(w, \mu, \Sigma)\sigma \tag{16}$$

is the CML of the least favorable asset return statistics and called the least favorable CML.

The minimax inequality or weak minimax property

$$\sup_{(\mu,\Sigma) \in \mathcal{U}} \inf_{w \in \mathcal{W}} S(w, \mu, \Sigma) \leq \inf_{(\mu,\Sigma) \in \mathcal{U}} \sup_{w \in \mathcal{W}} S(w, \mu, \Sigma)$$

holds for any uncertainty set $\mathcal{U}$. That is, the slope of the robust optimal CAL is no greater than that of the least favorable CML. As a consequence, when the inequality is strict, the portfolio that maximizes the worst-case SR is not the TP of any asset return statistics in $\mathcal{U}$.

The following proposition summarizes the minimax result for the zero-sum game mentioned above.

Proposition 3 (Saddle-point property of the SR [30])**.** *Suppose that the uncertainty set $\mathcal{U}$ is compact and convex, and the assumption (10) holds. Then, the SR satisfies the minimax equality*

$$\sup_{w \in \mathcal{W}} \inf_{(\mu,\Sigma) \in \mathcal{U}} S(w, \mu, \Sigma) = \inf_{(\mu,\Sigma) \in \mathcal{U}} \sup_{w \in \mathcal{W}} S(w, \mu, \Sigma). \tag{17}$$

Moreover, if the least favorable pair $(\mu^\star, \Sigma^\star)$ has the tangency portfolio $w^\star \in \mathcal{W}$, then the triple $(w^\star, \mu^\star, \Sigma^\star)$ satisfies the saddle-point property

$$S(w, \mu^\star, \Sigma^\star) \leq S(w^\star, \mu^\star, \Sigma^\star) \leq S(w^\star, \mu, \Sigma), \quad \forall\, w \in \mathcal{W}, \quad \forall\, (\mu, \Sigma) \in \mathcal{U}, \tag{18}$$

and $w^\star$ is the unique solution to the worst-case SRMP (13) although there may be multiple least favorable models.

This proposition tells us that when the uncertainty set $\mathcal{U}$ is convex, the two lines (15) and (16) coincide with each other. When the saddle-point property (18) holds, the slope of the robust optimal CAL can be written as

$$\sup_{w \in \mathcal{W}} \inf_{(\mu,\Sigma) \in \mathcal{U}} S(w, \mu, \Sigma) = \inf_{(\mu,\Sigma) \in \mathcal{U}} \sup_{w \in \mathcal{W}} S(w, \mu, \Sigma) = S(w^\star, \mu^\star, \Sigma^\star) = S_{\rm mp}(\mu^\star, \Sigma^\star).$$

With a convex uncertainty set $\mathcal{U}$, the worst-case MPRAP (14) can be reformulated as the convex optimization problem

$$\begin{array}{ll} \text{minimize} & (\mu - \mu_{\rm rf}\mathbf{1} + \lambda)^T \Sigma^{-1} (\mu - \mu_{\rm rf}\mathbf{1} + \lambda) \\ \text{subject to} & (\mu, \Sigma) \in \mathcal{U}, \quad \lambda \in \mathcal{W}^{\oplus}, \end{array}$$

in which the optimization variables are $\mu \in \mathbf{R}^n$, $\Sigma \in \mathbf{S}^n$, and $\lambda \in \mathbf{R}^n$. Here $\mathcal{W}^{\oplus}$ is the positive conjugate cone of $\mathcal{W}$,

$$\mathcal{W}^{\oplus} = \{\lambda \in \mathbf{R}^n \mid \lambda^T w \geq 0,\ \forall w \in \mathcal{W}\}.$$

The details are given in [30]. Since convex problems are tractable, we conclude that there is a general tractable method for computing the saddle point (if it exists).

The saddle-point property (18) for the SR has a geometric interpretation in the risk-return space. The risk-return set $\mathcal{P}(w^\star)$ of the portfolio $w^\star$ lies on or above the robust optimal CAL, which in turn lies on or above the efficient frontier of the least favorable asset return statistics $(\mu^\star, \Sigma^\star)$:

$$r \geq \mu_{\rm rf} + S(w^\star, \mu^\star, \Sigma^\star)\sigma \geq f_{\mu^\star,\Sigma^\star}(\sigma), \qquad \forall\ (\sigma, r) \in \mathcal{P}(w^\star).$$

The lower boundary of the set $\mathcal{P}(w^\star)$ and the efficient frontier of the least favorable asset return statistics $(\mu^\star, \Sigma^\star)$ meet at $(\sigma^\star, r^\star) = (r(w^\star, \mu^\star), \sigma(w^\star, \Sigma^\star))$:

$$r^\star = \mu_{\rm rf} + S(w^\star, \mu^\star, \Sigma^\star)\sigma^\star = f_{\mu^\star,\Sigma^\star}(\sigma^\star).$$

We conclude that the robust optimal CAL (15) is the CML of the least favorable asset return statistics $(\mu^\star, \Sigma^\star)$ and is tangential to the efficient frontier $f_{\mu^\star,\Sigma^\star}$ at the point $(\sigma^\star, r^\star)$. The robust optimal CAL is called the *robust CML*, and the portfolio that maximizes the worst-case SR is called the *robust tangency portfolio*. Figure 3 illustrates the geometric interpretation given above.

3.3 Robust optimal allocation and two-fund separation

We can observe from the definition of the robust optimal CAL that the set of possible risk-return pairs, consistent with the assumptions made on the model, of any admissible portfolio cannot lie entirely above the robust optimal CAL. The following proposition follows from the observation and (12).

Proposition 4 (Two-fund separation under model uncertainty)**.** *For any subset $\mathcal{U}$ of $\mathbf{R}^n \times \mathbf{S}^n_{++}$, the line (15) is the worst-case efficient frontier in the following sense:*

- *The risk-return set $\mathcal{P}((1-\theta)w, \theta)$ of any admissible portfolio $((1-\theta)w, \theta) \in \mathcal{X}$ cannot lie entirely above the robust optimal CAL, that is, there exists a point (σ, r) in $\mathcal{P}((1-\theta)w, \theta)$ such that*

$$r \leq \mu_{\rm rf} + \sup_{w \in \mathcal{W}} \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w, \mu, \Sigma)\sigma.$$

- *If the worst-case SRMP (13) has a solution $w^\star$, then for any $\theta < 1$, the risk-return set of the portfolio $((1-\theta)w^\star, \theta)$ lies above or on the robust optimal CAL:*

$$r \geq \mu_{\rm rf} + \sup_{w \in \mathcal{W}} \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w, \mu, \Sigma)\sigma, \quad (\sigma, r) \in \mathcal{P}((1-\theta)w^\star, \theta).$$

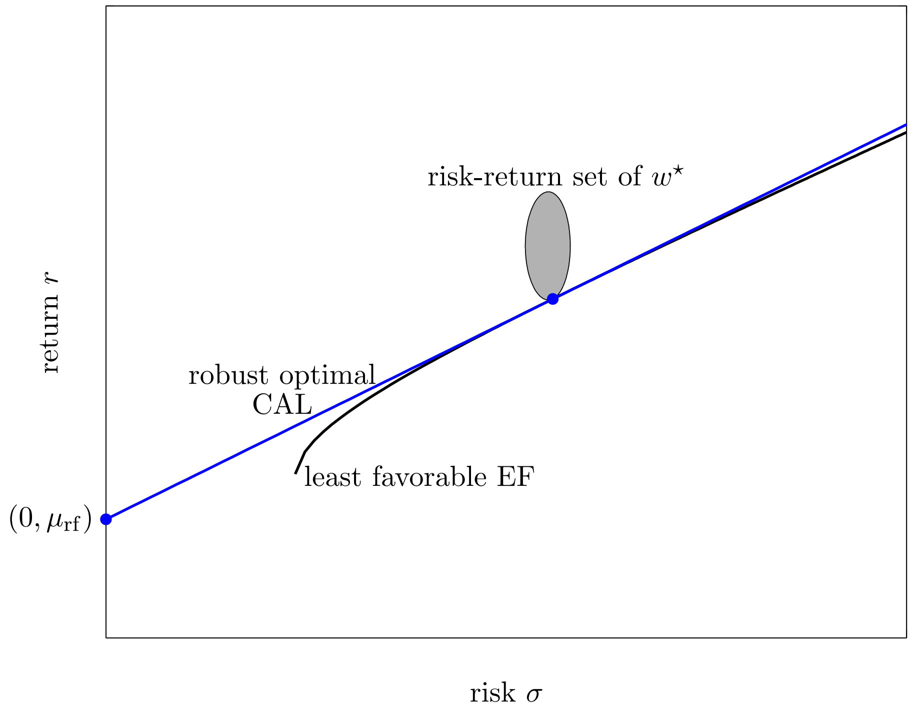

Figure 3: Saddle-point property of the Sharpe ratio.

We show how Proposition 4 leads to an extension of the classical two-step asset allocation process. The extension takes into account not only the attitude of the investor toward risk, described by the utility function, but also her attitude toward uncertainty, described by the set $\mathcal{U}$. We assume that the investor has ambiguity or uncertainty aversion, meaning that she would evaluate an investment strategy according to the expected utility under the worst case scenario in a set of model parameters. The investor would judge the performance of a portfolio x with the worst-case expected quadratic utility (over the uncertainty set $\mathcal{U}$)

$$U_{\rm wc}(x) = \inf_{(\mu,\Sigma)\in\mathcal{U}} U(x, \mu, \Sigma). \tag{19}$$

The investor would want to solve the robust counterpart of the expected quadratic utility problem (7), *i.e.*, the problem of finding the portfolio that maximizes the worst-case SR,

$$\begin{array}{ll} \text{maximize} & U_{\rm wc}(x) \\ \text{subject to} & x \in \mathcal{X}. \end{array} \tag{20}$$

The assumption (10) makes sense in worst-case expected quadratic utility maximization. If there is no admissible portfolio of risky assets whose worst-case mean return is greater than the return of the risk-free asset, the portfolio $(0, 1) \in \mathbf{R}^{n+1}$ consisting of only the risk-free asset maximizes the worst-case expected quadratic utility, and so an investor with quadratic utility and uncertainty aversion would invest only in the risk-free asset, regardless of her attitude toward risk.

The expected quadratic utility function is convex in (μ, Σ) over $\mathbf{R}^n \times \mathbf{S}^n_{++}$ for fixed x, and concave in x for fixed (μ, Σ). It follows from the standard minimax theorem for convex/concave functions that when $\mathcal{U}$ is convex and compact, the minimax equality

$$\sup_{x \in \mathcal{X}} \inf_{(\mu,\Sigma) \in \mathcal{U}} U(x, \mu, \Sigma) = \inf_{(\mu,\Sigma) \in \mathcal{U}} \sup_{x \in \mathcal{X}} U(x, \mu, \Sigma)$$

holds. From a standard result in minimax theory, when the worst-case EQUMP (20) has a solution, say $x^\star$, it along with the solution $(\mu^\star, \Sigma^\star)$ to the worst-case MPRAP (14) satisfies the saddle-point property

$$U(x, \mu^\star, \Sigma^\star) \le U(x^\star, \mu^\star, \Sigma^\star) \le U(x^\star, \mu, \Sigma), \quad \forall\, x \in \mathcal{X}, \quad \forall\, (\mu, \Sigma) \in \mathcal{U}. \tag{21}$$

The following proposition describes a closed-form solution to the worst-case EQUMP (20).

Proposition 5. *Suppose that the uncertainty set $\mathcal{U}$ is convex and compact, and the assumption (10) holds. Then, the worst-case EQUMP (20) has a solution if and only if the worst-case SRMP (13) has a solution. If $w^\star$ maximizes the worst-case SR, then the affine combination $x^\star = ((1-\theta^\star)w^\star, \theta^\star)$ of $w^\star$ and the risk-free asset, with*

$$\theta^\star = 1 - \frac{1}{\gamma} \frac{w^{\star T}\mu^\star - \mu_{\rm rf}}{w^{\star T}\Sigma^\star w^\star}, \tag{22}$$

is the unique solution to (20).

The proof is deferred to Appendix A.3.

This proposition tells us that due to her uncertainty aversion, the investor would hold a combination of the robust TP (when it exists) and the risk-free asset, regardless of her risk aversion. The fraction $\theta^\star$ is determined by her attitude toward risk (*i.e.*, the constant γ) and the risk-variance ratio of the robust TP when the asset return statistics are least favorable.

The saddle-point property (21) has a simple geometric interpretation in the risk-return space. The quadratic curve

$$r - (\gamma/2)\sigma^2 = U^\star, \tag{23}$$

with $U^\star = U((1-\theta^\star)w^\star, \theta^\star, \mu^\star, \Sigma^\star)$, is the optimal indifference curve when the asset return statistics are least favorable. We call this curve the robust optimal indifference curve. The saddle-point property means that the quadratic curve lies entirely above the robust CML except at the point $(\sigma^\star, r^\star) = (\sigma(x^\star, \Sigma^\star), r(x^\star, \mu^\star))$:

$$\mu_{\rm rf} + S_{\rm mp}(\mu^\star, \Sigma^\star)\sigma^\star = \frac{\gamma}{2}\sigma^{\star 2} + U^\star$$

and

$$\mu_{\rm rf} + S_{\rm mp}(\mu^\star, \Sigma^\star)\sigma < \frac{\gamma}{2}\sigma^2 + U^\star, \quad \sigma \ne \sigma^\star.$$

It also follows that the risk-return set of the portfolio $((1-\theta^\star)w^\star, \theta^\star)$ lies on or above the curve (23),

$$r - \frac{\gamma}{2}\sigma^2 \ge U^\star, \quad \forall\, (\sigma, r) \in \mathcal{P}((1-\theta^\star)w^\star, \theta^\star),$$

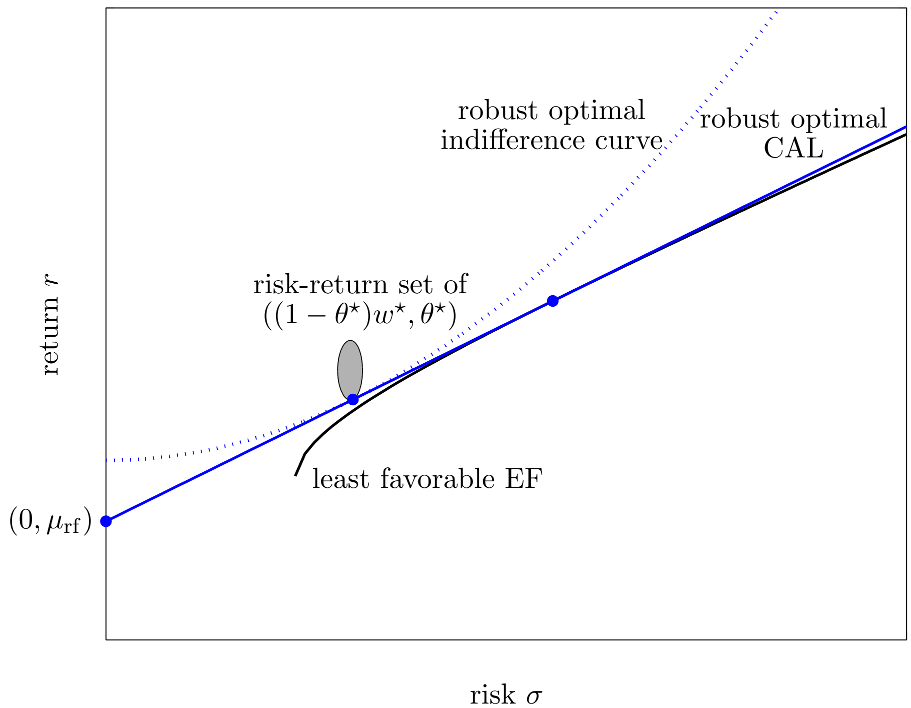

Figure 4: Two-fund separation under under model mis-specification and worst-case expected quadratic utility maximization.

and the lower boundary of the risk-return set and the curve meet at $(\sigma^\star, r^\star)$.

Figure 4 illustrates the extension of the theorem given above. (With the singleton $\mathcal{U} = \{(\mu, \Sigma)\}$, it reduces to the illustration of the classical two-fund separation theorem in figure 1).) The shaded region is the risk-return set $\mathcal{P}((1-\theta^\star)w^\star, \theta^\star)$ of the robust optimal portfolio $((1-\theta^\star)w^\star, \theta^\star)$ that maximizes the worst-case expected quadratic utility. The dotted curve corresponds to the robust optimal indifference curve $r = U^\star + (\gamma/2)\sigma^2$.

4 Numerical example

4.1 Setup and computation

We consider a synthetic example with 8 risky assets ($n = 8$) and

$$\mathcal{W} = \{w \in \mathbf{R}^n \mid \mathbf{1}^T w = 1\}.$$

The positive conjugate cone of $\mathcal{W}$ is

$$\mathcal{W}^\oplus = \{\eta \mathbf{1} \in \mathbf{R}^n \mid \eta \geq 0\}.$$

The risk-free return is $\mu_{\rm rf} = 4$.

The nominal returns $\bar{\mu}_i$ and nominal variances $\bar{\sigma}_i^2$ of the asset returns are taken as

$$\begin{aligned}\bar{\mu} &= (7.1, 6.9, 13.7, 11.0, 14.99, 10.4, 11.9, 14.7), \\ \bar{\sigma} &= (9.4, 8.1, 19.9, 14.4, 24.6, 15.7, 15.2, 27.8).\end{aligned}$$

All units here are in percentage. The nominal correlation matrix $\bar{\Omega}$ is taken as

$$\bar{\Omega} = \begin{bmatrix} 1 & .41 & .22 & .28 & .11 & .19 & .19 & .02 \\ & 1 & .03 & .06 & .08 & .14 & .39 & .11 \\ & & 1 & .69 & .82 & .58 & .62 & 0.65 \\ & & & 1 & .69 & .81 & .58 & 0.59 \\ & & & & 1 & .86 & .54 & 0.67 \\ & & & & & 1 & .50 & 0.62 \\ & & & & & & 1 & 0.71 \\ & & & & & & & 1 \end{bmatrix} \in \mathbf{R}^{8\times 8}.$$

(Only the upper triangular part is shown because the matrix is symmetric.) The risk-less return is $\mu_{\rm rf} = 3$. The nominal covariance is

$$\bar{\Sigma} = \mathrm{diag}(\bar{\sigma})\bar{\Omega}\,\mathrm{diag}(\bar{\sigma}),$$

where we use $\mathrm{diag}(z_1, \ldots, z_m)$ to denote the diagonal matrix with diagonal entries $z_1, \ldots, z_m$. The risk-less return of the risk-free asset is taken as $\mu_{\rm rf} = 3$. The nominal TP is the TP computed with the asset return statistics $(\bar{\mu}, \bar{\Sigma})$.

We now descrine the uncertainty set $\mathcal{U}$. We assume that the possible variation in the expected return of each asset is at most 20%:

$$|\mu_i - \bar{\mu}_i| \;\; \leq \;\; 0.2|\bar{\mu}_i|, \quad i = 1, \ldots, 7.$$

We also assume that the possible variation in each component of the covariance matrix is at most 20%:

$$|\Sigma_{ij} - \bar{\Sigma}_{ij}| \;\; \leq \;\; 0.2|\bar{\Sigma}_{ij}|, \quad i, j = 1, \ldots, 7,$$

and, of course, we require that $\Sigma \in \mathcal{S}$ be positive definite. We also assume that the variance and return of the uniform portfolio $\bar{w} = (1/n)\mathbf{1}$ (in which a fraction $1/n$ of budget is allocated to each asset of the n assets) is known to lie within an ellipse

$$\mathcal{E} = \left\{ (v, r) \in \mathbf{R}^2 \;\middle|\; (r - \bar{w}^T\bar{\mu})^2 + 0.01\left(v - \bar{w}^T\bar{\Sigma}\bar{w}\right)^2 \leq 1 \right\}$$

in the variance-return space.

The least favorable asset return statistics can be found by solving the convex problem

$$\begin{array}{ll} \text{minimize} & (\mu - \mu_{\rm rf}\mathbf{1} + \eta\mathbf{1})^T\Sigma^{-1}(\mu - \mu_{\rm rf}\mathbf{1} + \eta\mathbf{1}) \\ \text{subject to} & \eta \geq 0, \\ & (\bar{w}^T\Sigma\bar{w}, \bar{w}^T\mu) \in \mathcal{E}, \\ & |\Sigma_{ij} - \bar{\Sigma}_{ij}| \leq 0.2|\bar{\Sigma}_{ij}|, \quad i, j = 1, \ldots, n, \\ & |\mu_i - \bar{\mu}_i| \leq 0.2|\bar{\mu}_i|, \quad i = 1, \ldots, n, \end{array}$$

where $\mu \in \mathbf{R}^n$, $\Sigma = \Sigma^T \in \mathbf{R}^{n\times n}$, and $\eta \in \mathbf{R}$, are the variables. This problem can be reformulated as a semidefinite program, which interior-point methods can solve efficiently.

	nominal SR	worst-case SR
nominal TP	0.65	0.11
robust TP	0.58	0.48

Table 1: The nominal and worst-case SR of the two portfolios: nominal and robust tangency portfolios.

4.2 Numerical results

Table 1 shows the nominal and worst-case SR of the nominal optimal and robust optimal allocations. In comparison with the nominal optimal TP, the robust TP shows a relatively small decrease in the SR, in the presence of parameter variation. The SR of the robust TP decreases about 17% from 0.58 to 0.48, while the SR of the nominal TP decreases about 83% from 0.65 to 0.11. We see that the nominal performance of the robust TP is not too much worse to that of the nominal TP, but the robust TP is much more robust than the nominal TP to parameter variation.

Figure 5 compares the weights of the nominal and robust TPs. The nominal TP has short positions in some assets, while the robust TP has long positions in all assets. This figure shows that the nominal TP has some relatively large weights, which is one reason it is sensitive to variations in the parameters.

Figure 6 shows how the leverage of the risky portfolio varies as the constant γ varies. This proposition shows that uncertainty aversion reduces demand for the risky asset, which is in line with the result in [38].

Figure 7 compares the nominal expected quadratic utility, computed with the baseline model, achieved by the nominal optimal and robust optimal portfolios as γ varies. Since the nominal TP maximizes the SR for the baseline model, the combination of the robust TP and the risk-free asset cannot outperforms the combination of the nominal TP and the risk-free asset. Figure 8 compares the worst-case expected quadratic utility (EQU) achieved by the nominal optimal and robust optimal portfolios as γ varies. Since the robust TP maximizes the worst-case SR, the combination of the robust TP and the risk-free asset should outperforms the combination of the nominal TP and the risk-free asset, which is confirmed by this figure. The gap is especially large when the risk aversion constant γ is small. We can see a significant improvement brought about by the robust combination. Of course, the latter is less efficient than the nominal portfolio with the baseline model. Model uncertainty makes the nominal TP a poor choice over the robust TP.

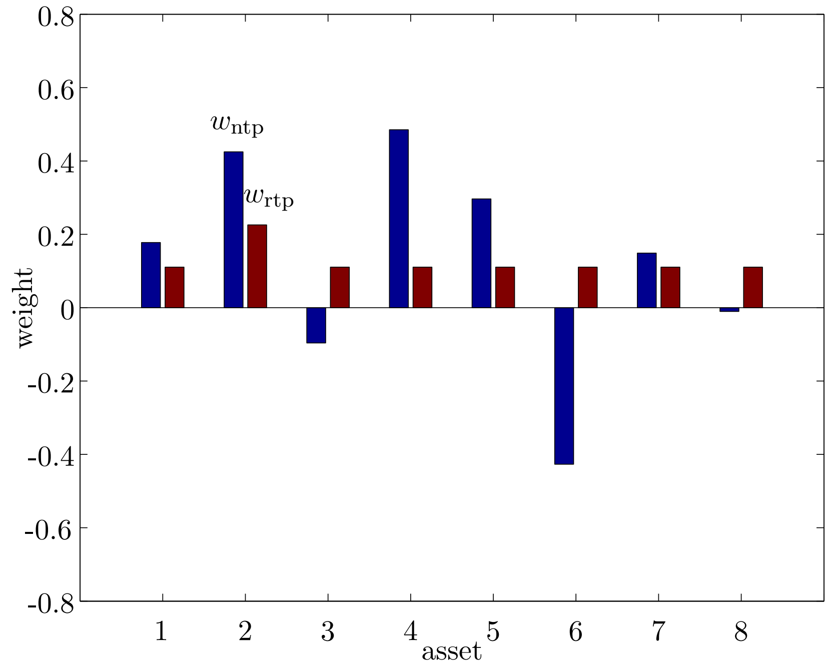

Figure 5: Weights of assets in nominal tangency portfolio and robust tangency portfolio.

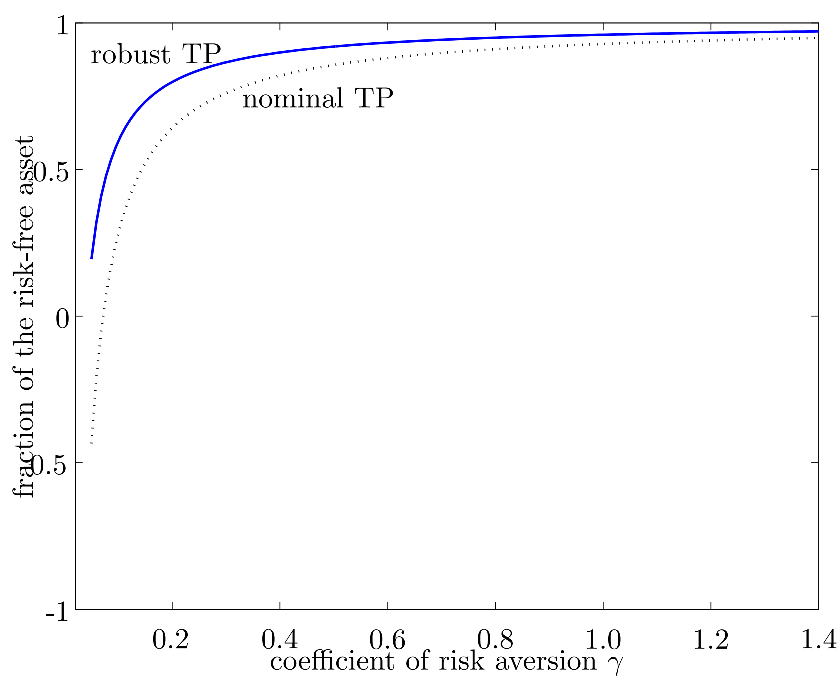

Figure 6: Fraction of the risk-free asset in the nominal optimal and robust optimal portfolios, depending on the coefficient of risk aversion.

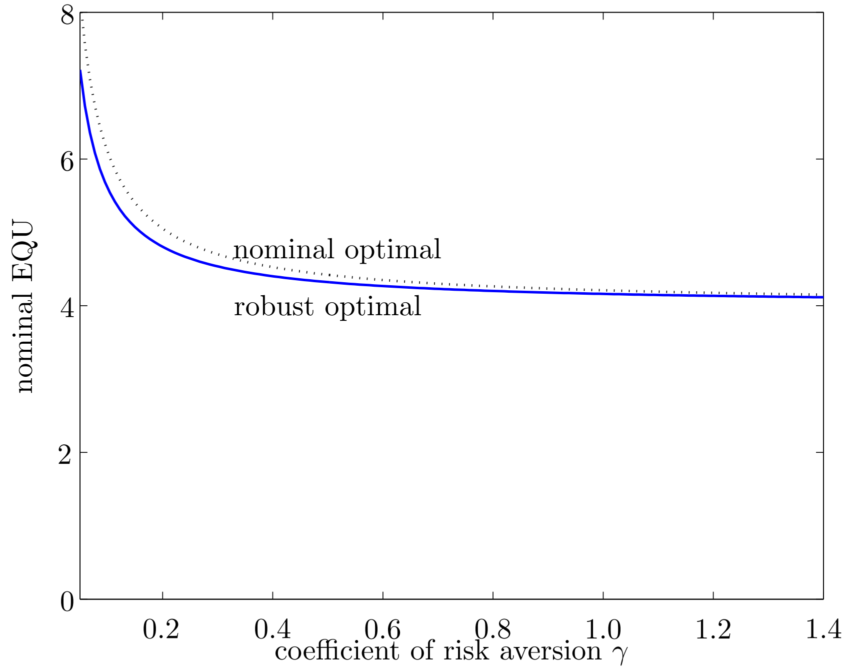

Figure 7: Nominal expected quadratic utilities of nominal optimal and robust optimal portfolios depending on the coefficient of risk aversion.

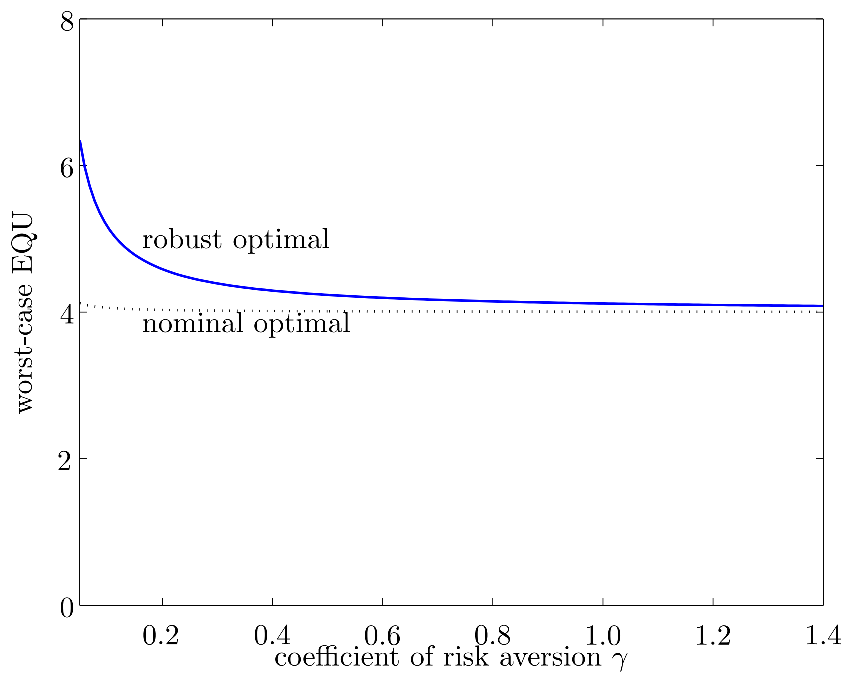

Figure 8: Worst-case expected quadratic utilities of nominal optimal and robust optimal portfolios depending on the coefficient of risk aversion.

5 Conclusions

In this paper, we have described an extension of the two-fund separation theorem to MV analysis under model mis-specification. The extension tells us that when the uncertainty model is convex, an investor with quadratic utility and uncertain aversion would hold a combination of the risk-free asset and the risky portfolio which is the tangency portfolio of the least favorable asset return statistics in terms of the market price of risk. The fraction is determined by her attitude toward risk, and the risky portfolio can be found efficiently using convex optimization.

The two-fund separation property holds for other several standard utility maximization problems than quadratic utility functions, when the asset returns are jointly normal; the reader is referred to [10, 2] for more on utility maximization problems compatible with the two-fund separation property. The two-fund separation property can be extended to certain types of worst-case utility maximization problems including exponential utility functions, when the asset returns are jointly normal. An exponential utility function has the form

$$U(c) = -e^{-\lambda c},$$

where λ is the coefficient of absolute risk aversion. When the asset returns are jointly normal, the expected exponential utility of a portfolio x is

$$\mathbf{E}U(x^T a) = -\exp\left(-(\mathbf{E}(x^T a) - (\lambda^2/2)\mathbf{V}(x^T a))\right),$$

which is an increasing function of the expected quadratic utility. Therefore, worst-case expected exponential utility maximization is the same as worst-case quadratic utility maximization with $\gamma = \lambda^2$, so the two-fund separation property readily extends. It is an interesting topic to clarify the class of worst-case utility maximization problems which exhibit the robust two-fund separation property. The two-fund separation property has also been extended to dynamic and other settings; see, *e.g.*, [41, 46] to name a few. It is also an interesting topic to extend the two-fund separation property to other settings while taking into account model mis-specification and uncertainty aversion.

The two-fund separation property has an important implication for equilibrium prices of assets, which is known as the CAPM. The extension of the two-fund separation theorem tells us that as long as investors with uncertainty aversion and quadratic utility have the same uncertainty model, they would hold a combination of the same portfolio of risky assets and the risk-free asset, regardless of their risk tolerance. An implication for equilibrium prices of assets is that under the standing assumptions of the CAPM and the additional assumption that all investors share the same convex uncertainty model, the robust TP that maximizes the worst-case SR is the market portfolio. An immediate observation we can make is that the market portfolio is not necessarily MV efficient when the true model is not least favorable. An interesting topic is to examine the implications of this observation in terms of uncertainty premium and build an asset pricing model which takes into account not only risk premium but also uncertainty premium. Related work in this direction includes [12, 33], which argue that equity premium can be decomposed into two components, risk premium and uncertainty premium.

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A Proofs

A.1 Convexity of the feasible asset allocation set $\mathcal{X}$

To establish the convexity, we must show that a convex combination $x = \alpha x^{(1)} + (1-\alpha)x^{(2)}$ of two admissible portfolios $x^{(1)} = ((1-\theta_1)w^{(1)}, \theta_1) \in \mathcal{X}$ and $x^{(2)} = ((1-\theta_2)w^{(2)}, \theta_2) \in \mathcal{X}$, where $\theta_1, \theta_2 \leq 1$ and $\alpha \in (0,1)$, belongs to $\mathcal{X}$. The portfolio x can be written as $x = ((1-\theta)w, \theta)$, where $\theta = \alpha\theta_1 + (1-\alpha)\theta_2 \leq 1$ and

$$w = \begin{cases} \dfrac{\alpha(1-\theta_1)}{1-\theta} w^{(1)} + \dfrac{(1-\alpha)(1-\theta_2)}{1-\theta} w^{(2)}, & \theta < 1 \\ 0, & \theta = 1. \end{cases}$$

Here, $\alpha(1-\theta_1), (1-\alpha)(1-\theta_2) \geq 0$. The case of $\theta = 1$ arise only when $w^{(1)} = (0,1)$ and $w^{(2)} = (0,1)$. In this case, $x = (0,1)$ is admissible, for any value of α. When $\theta < 1$, we have

$$\frac{\alpha(1-\theta_1)}{1-\theta} + \frac{(1-\alpha)(1-\theta_2)}{1-\theta} = \frac{\alpha(1-\theta_1)}{1-\alpha\theta_1-(1-\alpha)\theta_2} + \frac{(1-\alpha)(1-\theta_2)}{1-\alpha\theta_1-(1-\alpha)\theta_2} = 1.$$

In other words, w is a convex combination of $w^{(1)}, w^{(2)} \in \mathcal{W}$, so $x \in \mathcal{X}$.

A.2 Existence and uniqueness

Existence of the solution to the worst-case MPRAP (14)

Due to the portfolio budget constraint, we have

$$S(w, \mu, \Sigma) = \frac{w^T(\mu - \mu_{\mathrm{rf}})}{\sqrt{w^T \Sigma w}}, \quad w \in \mathcal{W}.$$

The worst-case MPRAP (14) is equivalent to

$$\begin{array}{ll} \text{minimize} & \sup_{w \in \mathcal{W}} \dfrac{w^T(\mu - \mu_{\mathrm{rf}})}{\sqrt{w^T \Sigma w}} \\ \text{subject to} & (\mu, \Sigma) \in \mathcal{U}. \end{array}$$

From Proposition 1 in [30], we can see that this problem (and hence (14)) has a solution.

Uniqueness of the solution to the worst-case SRMP (13)

Suppose that the worst-case SRMP (13) has two solutions $u^\star$ and $v^\star$, which are not identical. Due to the portfolio budget constraint, $u^\star$ and $v^\star$ are linearly independent. By the definition of the robust optimal CAL, the risk-return sets of $u^\star$ and $v^\star$ lie on and above, but cannot lie entirely above, the line in the (σ, r) space. Let $w^\star = (u^\star + u^\star)/2$. Using the Cauchy-Schwartz inequality, we can show that

$$\frac{1}{2}\left(\sqrt{u^{\star T}\Sigma u^\star} + \sqrt{v^{\star T}\Sigma v^\star}\right) > \sqrt{w^{\star T}\Sigma w^\star}$$

for any positive definite Σ, since $u^\star$ and $v^\star$ are linearly independent. We also have

$$w^{\star T}\mu - \mu_{\rm rf}\mathbf{1} = \frac{1}{2}\left(u^{\star T}\mu - \mu_{\rm rf}\mathbf{1} + v^{\star T}\mu - \mu_{\rm rf}\mathbf{1}\right).$$

The return of the portfolio $w^\star$ has the same return as the middle point of the line segment that connects the two points $(\sqrt{u^{\star T}\Sigma u^\star}, u^{\star T}\mu - \mu_{\rm rf}\mathbf{1})$ and $(\sqrt{v^{\star T}\Sigma v^\star}, v^{\star T}\mu - \mu_{\rm rf}\mathbf{1})$ but has a smaller risk. Since the line segment of any two points from the risk-return sets of $u^\star$ and $v^\star$ lies above the robust optimal CAL, we have that any point in the risk-return set of $w^\star$ lies strictly above the robust optimal CAL. Therefore,

$$\frac{r-\mu_{\rm rf}}{\sigma} > \sup_{w\in\mathcal{W}} \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w,\mu,\Sigma), \quad (\sigma, r) \in \mathcal{P}(w^\star).$$

It follows from the compactness of $\mathcal{U}$ that

$$\inf_{(\sigma,r)\in\mathcal{P}(w^\star)} \frac{r-\mu_{\rm rf}}{\sigma} > \sup_{w\in\mathcal{W}} \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w,\mu,\Sigma),$$

which contradicts the definitions of $u^\star$ and $v^\star$ as the solutions of the worst-case SRMP (13). We conclude that if the worst-case SRMP (13) has a solution, then it must be unique.

A.3 Proof of Proposition 5

We start by observing that, in the variance-return space, the worst-case expected quadratic utility of x can be expressed as

$$U_{\rm wc}(x) = \inf\{r - (\gamma/2)v \mid (v,r) \in \mathcal{Q}(x)\},$$

where $\mathcal{Q}(x)$ is the set of possible pairs of variance and return of x (over the uncertainty set $\mathcal{U}$),

$$\mathcal{Q}(x) = \{(\sigma(x,\Sigma)^2, r(x,\mu)) \in \mathbf{R}^2 \mid (\mu,\Sigma) \in \mathcal{U}\}.$$

The set $\mathcal{Q}(x)$ is convex, since the variance and return of x are linear in the mean return and covariance.

In the variance-return space, the worst-case CAL of a portfolio w is transformed into the strictly concave curve

$$h(v) = \mu_{\rm rf} + S_{\rm wc}(w)\sqrt{v}.$$

There is only one line with slope $\gamma/2$ which is tangential to the curve $r = h(v)$, $r = (\gamma/2)v + \bar{U}$, where $\bar{U}$ is the return-intercept of the line. The tangential point corresponds to the worst-case risk-return pair $(\bar{\sigma}, \bar{r})$ of a portfolio $\bar{x} = (1-\bar{\theta})w, \bar{\theta}$ with $\bar{\theta} < 1$.

We show that

$$\bar{U} = \sup_{\theta<1} U_{\rm wc}((1-\theta)w, \theta), \tag{24}$$

and the supremum is uniquely achieved by the portfolio $\bar{x} = (1-\bar{\theta})w, \bar{\theta}$. In the variance-return space, indifference curves with the same utility are lines with slope $\gamma/2$. The risk-return set of any admissible portfolio, which is convex in the variance-return space, cannot lie above the curve $r = h(v)$. Since there exists a point (v, r) in $\mathcal{Q}((1-\theta)w, \theta)$ such that $r \le h(v)$, we have

$$r \le h(v) \le \frac{\gamma}{2}v + \bar{U},$$

so $r - (\gamma/2)v \le \bar{U}$. Therefore, for any $x = ((1-\theta)w, \theta)$ with $\theta < 1$,

$$U_{\rm wc}(x) = \inf_{(\mu,\Sigma)\in\mathcal{U}} U(x, \mu, \Sigma) = \inf\{r - (\gamma/2)v \mid (v, r) \in \mathcal{Q}(x)\} \le \bar{U}.$$

We next note that

$$\frac{\gamma}{2}v + \bar{U} \ge h(v), \qquad \forall\,(\sigma, r) \in \mathcal{Q}(\bar{x})$$

and

$$\bar{r} = \frac{\gamma}{2}\bar{v} + \bar{U} = h(\bar{v}), \tag{25}$$

where $\bar{v} = \bar{\sigma}^2$. The set $\mathcal{Q}(x)$ is convex, and so its lower boundary is convex. A simple argument shows that

$$r \ge \frac{\gamma}{2}v + \bar{U}, \qquad \forall\,(\sigma, r) \in \mathcal{Q}(x)$$

To sum up, we have

$$r \ge \frac{\gamma}{2}v + \bar{U} \ge h(v), \qquad \forall\,(\sigma, r) \in \mathcal{Q}(x) \tag{26}$$

From (25) and (26), we can also show that the worst-case utility of $\bar{x}$ is $\bar{U}$:

$$\inf_{(\mu,\Sigma)\in\mathcal{U}} U(x, \mu, \Sigma) = \inf\{r - (\gamma/2)v \mid (v, r) \in \mathcal{Q}(x^\star)\} = \bar{U}.$$

As a consequence of (24), we have

$$0 < S_{\rm wc}(w) < S_{\rm wc}(\bar{w}) \qquad \Longrightarrow \qquad \sup_{\theta<1} U_{\rm wc}((1-\theta)w, \theta) < \sup_{\theta<1} U_{\rm wc}((1-\theta)\bar{w}, \theta). \tag{27}$$

This implication can be seen by noting that

$$\mu_{\rm rf} + S_{\rm wc}(w)\sqrt{v} < \mu_{\rm rf} + S_{\rm wc}(\bar{w})\sqrt{v}, \quad v > 0,$$

so the return-intercept of the line with slope $\gamma/2$ which is tangential to the curve $r = \mu_{\rm rf} + S_{\rm wc}(\bar{w})\sqrt{v}$ is larger than that of the line with the same slope which is tangential to the curve $r = \mu_{\rm rf} + S_{\rm wc}(w)\sqrt{v}$.

We consider the case when the worst-case SRMP (13) has a unique solution $w^\star$. In the variance-return space, the robust CML is transformed into the strictly concave curve

$$h^\star(v) = \mu_{\rm rf} + \sup_{w\in\mathcal{W}} \inf_{(\mu,\Sigma)\in\mathcal{U}} S(w, \mu, \Sigma)\sqrt{v} = \mu_{\rm rf} + S_{\rm mp}(\mu^\star, \Sigma^\star)\sqrt{v}.$$

It follows from Proposition 4 that there is a unique $\theta^\star < 1$ such that the tangential point on the curve $r = h^\star(v)$ corresponds to the risk $\sigma^\star = \sigma(x^\star, \Sigma^\star)$ and the return $r^\star = r(x^\star, \mu^\star)$ of a portfolio $x^\star = ((1-\theta)w^\star, \theta^\star)$:

$$r \ge \frac{\gamma}{2}v + U^\star \ge h^\star(v), \qquad \forall\,(\sigma, r) \in \mathcal{Q}(x^\star) \tag{28}$$

and

$$r^\star \geq \frac{\gamma}{2} v^\star + U^\star \geq h^\star(v^\star), \tag{29}$$

where $v^\star = \sigma^{\star 2}$. It follows from (27) that $x^\star = ((1-\theta^\star)w^\star, \theta^\star)$ solves the worst-case EQUMP (20). Since the solution of the worst-case SRMP (13) is unique, we can see that (20) has also a unique solution. Moreover, the saddle-point property follows from (28) and (29).

We next turn to the case when the worst-case SRMP (13) has no solution. Suppose that the worst-case EQUMP (20) has a solution, say $x^\star = ((1-\theta^\star)w^\star, \theta^\star)$. Then, there is $\bar{w} \in \mathcal{W}$ such that $S_{\mathrm{wc}}(w^\star) < S_{\mathrm{wc}}(\bar{w})$ and hence

$$\sup_{\theta<1} U_{\mathrm{wc}}((1-\theta)w^\star, \theta) < \sup_{\theta<1} U_{\mathrm{wc}}((1-\theta)\bar{w}, \theta),$$

which along with (27) shows that $x^\star$ cannot be a solution to (20). We conclude that the worst-case EQUMP (20) has no solution.

We complete the proof by deriving the formula (22) for the optimal ratio. The derivative of $h^\star$ at $v^\star$ is equal to the return value of the tangential point $(v^\star, \mu_{\mathrm{rf}} + S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)\sqrt{v^\star})$, that is, $S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)/2\sqrt{v} = \gamma/2$, where $S_{\mathrm{mp}}(\mu^\star, \Sigma^\star) = (w^{\star T}\mu^\star - \mu_{\mathrm{rf}})/\left(w^{\star T}\Sigma^\star w^\star\right)^{1/2}$. Therefore, $v^\star = \left(S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)/\gamma\right)^2$, so the tangential point is $\left(S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)/\gamma, \mu_{\mathrm{rf}} + S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)^2/\gamma\right)$. The return at the tangential point satisfies the equation

$$(1-\theta^\star)w^{\star T}\mu^\star + \theta^\star \mu_{\mathrm{rf}} = \mu_{\mathrm{rf}} + S_{\mathrm{mp}}(\mu^\star, \Sigma^\star)^2/\gamma.$$

We solve the equation to obtain (22).