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Subharmonic Functions and Performance Bounds on Linear Time-Invariant Feedback Systems

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In this paper we study *multiple-input multiple-output* (MIMO) *linear time-invariant* (LTI) control systems. We show that some well known constraints on the performance of *single-input single-output* (SISO) linear control systems, e.g. those expressed by the Paley–Wiener theorem, Bode's integral theorem, and more recently, Zames' inequality can be given a unified treatment using some elementary properties of *subharmonic functions*. Most importantly, results derived in this framework of subharmonic functions apply immediately to the MIMO case. Indeed the proofs of the MIMO generalizations are often simpler than the original proofs of the SISO versions.

1. Introduction

BODE (1945) was perhaps the first to study *a priori* constraints on the performance of single-input single-output (SISO) linear time invariant feedback systems, in the context of feedback amplifiers; Horowitz (1963) later interpreted Bode's work for control systems. In fact the classic Paley–Wiener theorem (1934) can also be interpreted as expressing an *a priori* constant on the performance of control systems. Recently this topic of deriving constraints on control system performance, based on just a little qualitative knowledge of the plant and controller (e.g. closed-loop stability and the existence of a plant $\mathbb{C}_+$ zero) has received much attention, for example in Zames (1981, 1983) and Freudenberg and Looze (1983).

The constraints arise from the requirement that the closed-loop system be stable, that is, that certain transfer functions be analytic and bounded in the right half-plane. The results mentioned above are all proved using the theory of *analytic functions* and as a result their multiple-input multiple-output (MIMO) analogues either do not exist or involve complicated proofs when they do.

One approach to extend these ideas to MIMO control systems focuses on the *eigenvalues* of the various transfer-function matrices: the eigenvalues are viewed as one *algebraic function* defined on a Reimann surface (Doyle and Stein, 1981). Unfortunately, the eigenvalues can be a poor measure of the "size" of a MIMO operator (e.g. a disturbance-to-output map) or the "tightness" of a feedback loop (Doyle and Stein, 1981). Matrix norms, for example the largest singular value, are good indicators of the size (resp., minimum singular values for the "tightness" of a feedback loop), but, to quote Looze and Freudenberg (1983): "In contrast to the gain of a scalar transfer function, a singular value is not in general the magnitude of an analytic function, thus precluding the application of complex variable theory which led to the Bode gain–phase relations."

While the maximum singular value of a stable transfer function matrix is not the real part of an analytic function (such functions are called *harmonic*), we will see that it is *subharmonic*, and that subharmonic functions have the properties needed to derive the constraints (or generalizations) above. The purpose of our paper is to show that using some elementary properties of subharmonic functions, all of the results mentioned above can be easily and clearly extended to the MIMO case. The proofs based on subharmonic functions not only apply to MIMO systems, but are often simpler than the original proofs of the SISO versions.

The mathematics presented here (Theorems 2.1 and 2.2) is *not*, to our knowledge, in the mathematics literature.

2. Subharmonic functions

2.1 *Notation and Definition*

$\mathbb{C}_+$ will denote the open right half plane $\{s : \operatorname{Re} s > 0\}$ and $\overline{\mathbb{C}_+}$ its closure $\{s : \operatorname{Re} s \geq 0\}$. H^∞ will denote as usual the set of functions $h(s)$ analytic and bounded in $\mathbb{C}_+$, with boundary values defined via

$$h(j\omega) = \lim_{\sigma \to 0} h(\sigma + j\omega). \tag{2.1}$$

(The limit in (2.1) can be shown to exist for almost all $\omega \in \mathbb{R}$: see e.g. Rudin (1984).

$(H^\infty)^{m \times n}$ will denote the set of $m \times n$ matrices with elements in H^∞. If A is an $m \times n$ complex matrix, then $\|A\|$ will denote any induced norm, for example the maximum singular value $\sigma_{\max}(A) \triangleq \sqrt{[\lambda_{\max}(A^*A)]}$.

We will be considering functions on $\mathbb{C}_+$ such as $f(s) = \log|(s-1)/(s+1)|$ which may take on the value $-\infty$, that is, functions $f : \mathbb{C}_+ \to [-\infty, \infty)$. Such a function is said to be *continuous* if the (real valued) function $\exp f$ is. This agrees with the standard notion of continuity when, as usual, a basis for the neighbourhoods of $-\infty$ are $[-\infty, -n)$ $(n = 1, 2, \ldots)$. Alternatively, *continuity* can be replaced in the sequel by *upper semicontinuity*, which is all that is needed.

DEFINITION $f : \mathbb{C}_+ \to [-\infty, \infty)$ is *subharmonic* if and only if it is continuous and whenever $\operatorname{Re} a > r > 0$

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{j\theta})\, d\theta. \tag{2.2}$$

Note that the continuity of f implies that the integrand in (2.2) is bounded above, and thus the integral in (2.2) is always well defined (but may be $-\infty$). Note also that the constant $-\infty$ is subharmonic. Finally, if equality always holds in (2.2), then f is *harmonic*.

We will be concerned with subharmonic functions which have a few additional properties. For notational convenience we define the class SH:

DEFINITION: $f:\overline{\mathbb{C}_+} \rightarrow [-\infty, \infty) \in \mathrm{SH}$ if and only if

(i) f is subharmonic,
(ii) f is bounded above, and
(iii) $\lim_{\sigma \to 0} f(\sigma + j\omega)$ exists and equals $f(j\omega)$ for almost all $\omega \in \mathbb{R}$.

2.2. SH *Functions in Control Theory*

Classic examples of functions in SH are $|h(s)|$ and $\log|h(s)|$, where $h(s) \in \mathrm{H}^\infty$. Some important SH functions in control theory are given in the following theorems.

THEOREM 2.1 *Suppose* $H(s) \in (\mathrm{H}^\infty)^{m \times n}$. *If* $\|\bullet\|$ *is any induced norm, then*

$$\|H(s)\| \in \mathrm{SH} \quad \textit{and} \quad \log\|H(s)\| \in \mathrm{SH}.$$

THEOREM 2.2 *Suppose* $H(s) \in (\mathrm{H}^\infty)^{n \times n}$. *Then*

$$\rho(H(s)) \in \mathrm{SH} \quad \text{and} \quad \log \rho(H(s)) \in \mathrm{SH}$$

where $\rho(A)$ *denotes the spectral radius of* A, *that is*, $\rho(A) \triangleq \max_i |\lambda_i(A)|$.

$$\mu(H(s)) \in \mathrm{SH} \quad \text{and} \quad \log \mu(H(s)) \in \mathrm{SH}$$

where μ *is Doyle's* structured singular value (Doyle, 1982). *If in addition* $H(s)^{-1} \in (\mathrm{H}^\infty)^{n \times n}$, *then*

$$\mathrm{cond}_{\|}(H(s)) \in \mathrm{SH} \quad \text{and} \quad \log \mathrm{cond}_{\|}(H(s)) \in \mathrm{SH}$$

where $\mathrm{cond}_{\|}A = \|A\|\,\|A^{-1}\|$ *is the condition number of* A.

Theorems 2.1 and 2.2 are proved in the appendix. We will not use the fact that Doyle's structured singular value and the condition number are SH in the sequel.

2.3 *Important Properties of* SH *Functions: MIMO Paley–Wiener Theorem*

In the sequel we will use only two properties of SH functions, the *Maximum principle* and the *Poisson inequality*.

MAXIMUM PRINCIPLE If $f \in \mathrm{SH}$ then

$$\sup_{\omega \in R} f(j\omega) = \sup_{\mathrm{Re}\, s \geq 0} f(s). \tag{2.3}$$

Technically, the sup on the left hand side of (2.3) is an essential sup. The proof can be found in Rudin (1974, p. 231) or Conway (1978, p. 264).

Remark. The Maximum principle need not hold if f is not bounded *above*, for example $f(s) = \mathrm{Re}\,(e^s)$, which is bounded along the jω-axis but not in $\mathbb{C}_+$. This f satisfies (i) and (iii) of SH but not (ii). Note also that the maximum principle still holds even if f is not bounded *below*, e.g. $f(s) = \log|(s-1)/(s+1)|$ which is in SH.

POISSON INEQUALITY Suppose $f \in \mathrm{SH}$ and is not identically $-\infty$. Then for $\sigma_0 > 0$,

$$\frac{1}{\pi}\int |f(\mathrm{j}\omega)| \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2 + (\omega - \omega_0)^2} < \infty \quad \text{i.e.} \quad \omega \to \frac{\sigma_0 f(\mathrm{j}\omega)}{\sigma_0^2 + (\omega + \omega_0)^2} \in \mathrm{L}^1, \tag{2.4}$$

and

$$\frac{1}{\pi}\int f(\mathrm{j}\omega) \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2 + (\omega - \omega_0)^2} \geq f(\sigma_0 + \mathrm{j}\omega_0). \tag{2.5}$$

Remark. Equation (2.5) should be compared to the *Poisson formula*, valid for a *bounded harmonic* function $f(s)$: for $\sigma_0 > 0$,

$$\frac{1}{\pi}\int f(\mathrm{j}\omega) \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2 + (\omega - \omega_0)^2} = f(\sigma_0 + \mathrm{j}\omega_0). \tag{2.6}$$

SISO arguments which use (2.6) (perhaps implicitly, e.g. via the Bode gain–phase relations) can be extended to analogous MIMO arguments simply by using the Poisson inequality in place of the Poisson formula. In this way SISO results go through nearly unchanged for the MIMO case, with the conclusion changed into the appropriate *inequality*.

Proof of Poisson inequality. Define $f_n(\mathrm{j}\omega) \triangleq \max\{f(\mathrm{j}\omega), -n\}$. Hence $\omega \to f_n(\mathrm{j}\omega)$ is bounded on $\mathbb{R}$. For $x > 0$ define

$$f_n(x + \mathrm{j}y) \triangleq \frac{1}{\pi}\int f_n(\mathrm{j}\omega) \frac{x\,\mathrm{d}\omega}{x^2 + (\omega - y)^2}. \tag{2.7}$$

(Note that the integrand in (2.7) is L^1.) $f_n : \overline{\mathbb{C}_+} \to \mathbb{R}$ is a *bounded harmonic function* which satisfies property (iii) of SH, so $f - f_n \in \mathrm{SH}$. Since $f(\mathrm{j}\omega) - f_n(\mathrm{j}\omega) \leq 0$ for all $\omega \in \mathbb{R}$, we conclude from the Maximum principle that $f(s) - f_n(s) \leq 0$ for all $s \in \mathbb{C}_+$. Thus for all n,

$$\frac{1}{\pi}\int f_n(\mathrm{j}\omega) \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2 + (\omega - \omega_0)^2} \geq f(\sigma_0 + \mathrm{j}\omega_0). \tag{2.8}$$

We now establish (2.4). If we can establish (2.4) for any particular $\sigma_0 = x > 0$ and $\omega_0 = y \in \mathbb{R}$, then it is true for all $\sigma_0 > 0$ and $\omega_0 \in \mathbb{R}$, since for each such σ_0 and ω_0 there is a $K < \infty$ such that for all $\omega \in \mathbb{R}$,

$$\frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \leq K \frac{x}{x^2 + (\omega - y)^2}$$

and hence

$$\int |f(\mathrm{j}\omega)| \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2 + (\omega - \omega_0)^2} \leq K \int |f(\mathrm{j}\omega)| \frac{x\,\mathrm{d}\omega}{x^2 + (\omega - y)^2}.$$

Since f is not identically $-\infty$, find $x>0$ and $y \in \mathbb{R}$ such that $f(x+jy) \neq -\infty$. Since $f \in \mathrm{SH}$ it is bounded above, we can find an $M<\infty$ such that $f(s) \leqslant M$ for all $s \in \overline{\mathbb{C}_+}$. Then

$$\int_{f_n(j\omega)\geqslant 0} |f_n(j\omega)| \frac{x\,d\omega}{x^2+(\omega-y)^2} \leqslant \pi M. \tag{2.9}$$

Now for each n we have:

$$\begin{aligned}
&\int |f_n(j\omega)| \frac{x\,d\omega}{x^2+(\omega-y)^2}\\
&\quad = 2\int_{f_n(j\omega)\geqslant 0} |f_n(j\omega)| \frac{x\,d\omega}{x^2+(\omega-y)^2} - \int f_n(j\omega) \frac{x\,d\omega}{x^2+(\omega-y)^2}\\
&\quad \leqslant 2\pi M - \pi f(x+jy)
\end{aligned}$$

by (2.8) and (2.9). Thus by the monotone convergence theorem (Rudin, 1974, p. 22)

$$\int |f(j\omega)| \frac{x\,d\omega}{x^2+(\omega-y)^2} \leqslant 2\pi M - \pi f(x+jy) < \infty$$

and (2.4) is established.

From (2.4), (2.8), and the dominated convergence theorem (Rudin, 1974, p. 27) we conclude

$$\frac{1}{\pi}\int f(j\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2+(\omega-\omega_0)^2} \geqslant f(\sigma_0+j\omega_0)$$

which is (2.5). □

Condition (2.4) implies that the map $\omega \to f(\omega)$ is locally L^1, which is not obvious at all. For example, $f(s)=\log|s(s+1)^{-1}|$ has a singularity at $\omega=0$, nevertheless is still locally L^1. Indeed (2.4) has as a corollary the following theorem.

MIMO PALEY–WIENER THEOREM. *Suppose $H(s) \in (H^\infty)^{m\times n}$ and is not identically zero. Then*

$$\int \frac{|\log \|H(j\omega)\||}{1+\omega^2}\,d\omega < \infty.$$

This is the simplest proof of the Paley–Wiener theorem that we know of (see e.g. Zadeh and Desoer (1963)).

3. Applications: MIMO feedback systems

3.1. *Set-up and Notation*

We will refer to the system ${}^1S(P, C)$ shown in Fig. 1. In order to include *distributed* and *unstable* plant $P(s)$ and compensator $C(s)$, we assume that $P(s)$

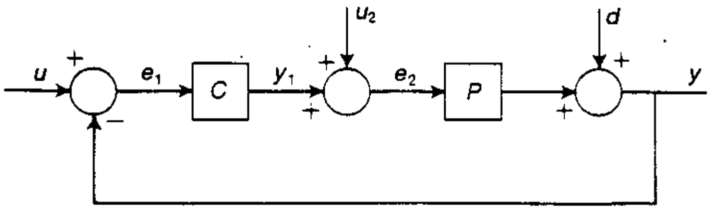

FIG. 1

(respectively, $C(s)$) is an $n \times m$ (resp., $m \times n$) matrix of transfer functions in the algebra B, with $n \leq m$ (see Callier and Desoer (1980) and Desoer and Vidysagar (1975)).

B is defined as follows: A is the subalgebra of H^{∞} consisting of Laplace transforms of distributions of the form

$$f(t) = f_a(t) + \sum_{i=0}^{\infty} f_i \delta(t - t_i)$$

where $f_a(t) = 0$ for $t < 0$, $t_i \geq 0$, $f_a \in L^1$, and $(f_i) \in \ell^1$. A^{∞} is the multiplicative subset consisting of those elements in A which are bounded away from zero at $s = \infty$ in $\mathbb{C}_+$. Finally, B is the algebra of quotients $A(A^{\infty})^{-1}$, that is, elements of B have the form n/d with $n \in A$ and $d \in A^{\infty}$ (see e.g. Lang, 1965, p. 66). The reader unfamiliar with these concepts can simply think of P and C as rational.

An element of B is *stable* if it is in A, i.e. if it has a representation with $d = 1$. We say an element h of B is *strictly proper* if $\lim_{s \to \infty} h(s) = 0$ in $\mathbb{C}_+$.

We make two assumptions about ${}^1S(P, C)$:

ASSUMPTION 1 The plant $P(s)$ is strictly proper.

ASSUMPTION 2 ${}^1S(P, C)$ is closed-loop stable, that is,

$$H_{eu} = \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix} \in A^{(m+n)\times(m+n)} \tag{3.1}$$

where $H_{eu} : (\boldsymbol{u}_1, \boldsymbol{u}_2) \to (\boldsymbol{e}_1, \boldsymbol{e}_2)$. This is precisely the condition that the transfer function from any input to any output has all its elements in A (and hence H^{∞}).

P and C have left and right A-coprime factorizations (Callier and Desoer, 1980, and Callier, Chan and Desoer, 1978):

$$P = D_{PL}^{-1} N_{PL} = N_{PR} D_{PR}^{-1},$$
$$C = D_{CL}^{-1} N_{CL} = N_{CR} D_{CR}^{-1},$$

with the N's and D's having elements in A and the D's having determinants in A^{∞}, e.g. $\det D_{PR} \in A^{\infty}$. The $\overline{\mathbb{C}_+}$ poles of P and C are precisely the $\overline{\mathbb{C}_+}$ zeros of $\det D_{PR}$ and $\det D_{CL}$, respectively.

We will say that $P(s)$ has a *zero* at $s_0 \in \overline{\mathbb{C}_+}$ if $N_{PL}(s_0)$ is less than full rank, that is, if there is a nonzero $\boldsymbol{c} \in \mathbb{C}^n$ such that $\boldsymbol{c}^T N_{PR}(s_0) = 0$. This agrees with the standard notion of a zero for rational P, defined via the Smith–MacMillan form (Callier and Desoer, 1982). Note that P may also have a *pole* at s_0. We define the *left nullspace* associated with the zero at s_0 by

$$\mathcal{N}_{\text{zero}} \triangleq \text{nullspace}\,(N_{PR}^{\mathrm{T}}(s_0));$$

$\mathcal{N}_{\text{zero}}$ does not depend on the coprime factorization used. If P does not have a pole at s_0, then P is analytic at s_0 and we can find $\mathcal{N}_{\text{zero}}$ without reference to coprime factorizations: $\mathcal{N}_{\text{zero}} = \text{nullspace}(P(s_0)^{\mathrm{T}})$. An example where $\mathcal{N}_{\text{zero}} \neq \text{nullspace}(P(s_0)^{\mathrm{T}})$ is:

$$P(s) = \frac{1}{(s+1)^2}\begin{bmatrix} s-2 & 0 \\ 1 & (s-2)^{-1} \end{bmatrix}.$$

For this plant there is no nonzero $\boldsymbol{\lambda} \in \mathbb{C}^2$ such that $\boldsymbol{\lambda}^{\mathrm{T}} P(s)$ is analytic and zero at $s = 2$.

Similarly if p_0 is a pole of P, we define its associated *right nullspace* $\mathcal{N}_{\text{pole}}$ by

$$\mathcal{N}_{\text{pole}} \triangleq \text{nullspace}\,(D_{PL}\,(p_0)).$$

$\mathcal{N}_{\text{pole}}$ is independent of the coprime factorization used, and if the pole is simple, then $\mathcal{N}_{\text{pole}}$ is also given by

$$\mathcal{N}_{\text{pole}} = \text{range}\,(\text{residue}_{p_0}\,P).$$

$\mathcal{N}_{\text{pole}}$ can be interpreted as those directions along which the (open-loop) plant blows up at p_0.

We will focus our attention on constraints imposed on the *disturbance-to-output map* $H_{yd} = (I + PC)^{-1}$, which also happens to be the *input-to-tracking-error* map $\boldsymbol{u} \to \boldsymbol{u} - \boldsymbol{y} = \boldsymbol{e}_1$. By either interpretation it is something which, roughly speaking, we would like "small" over the bandwidth of our system. We will refer to $-\log \|H_{yd}(j\omega)\|$ as the *disturbance rejection* (in nepers, at ω rad/sec). Of course other control configurations or transfer functions can be considered.

In the sequel we will use only a few properties of ${}^1S(P, C)$. Two very important ones relate to H_{yd}:

FACT. Suppose P has a zero at $s_0 \in \mathbb{C}_+$ and a pole at $p_0 \in \mathbb{C}_+$, with associated left (right) nullspace $\mathcal{N}_{\text{zero}}$ ($\mathcal{N}_{\text{pole}}$). Then
(1) If $\boldsymbol{\lambda} \in \mathcal{N}_{\text{zero}}$ then $\boldsymbol{\lambda}^{\mathrm{T}} H_{yd}(s_0) = \boldsymbol{\lambda}^{\mathrm{T}}$. In particular, $\|H_{yd}(s_0)\| \geqslant 1$.
(2) If $\boldsymbol{\mu} \in \mathcal{N}_{\text{pole}}$ then $H_{yd}(p_0)\boldsymbol{\mu} = 0$.

The proof is in §A2.

Interpretation of (1): At $s = s_0$, the component of the disturbance which lies in $\mathcal{N}_{\text{zero}}$ appears unaffected in the output.

Interpretation of (2): At $s = p_0$, we have perfect tracking along those directions in which the plant has infinite gain.

We have already seen one constraint on H_{yd}: by Assumption 1, PC is not identically $-I$, so H_{yd} cannot be identically zero, hence the Paley–Wiener theorem yields

$$\int \frac{|\log \|H_{yd}(j\omega)\||}{1+\omega^2}\,d\omega < \infty. \tag{3.2}$$

This constraint is well known in the SISO case. One consequence is, *we cannot have perfect tracking or infinite disturbance rejection* ($H_{yd}(j\omega) = 0$) *over any band of frequencies.*

3.2 *Bode's Integral*

In (3.2), the Paley–Wiener theorem expresses a fundamental constraint on the achievable disturbance rejection. If the plant and compensator are *strictly proper*, as is usually the case, the conclusion can be strengthened considerably. For SISO systems with *stable rational* P and C, with $P(s)C(s) = O(s^{-2})$, Bode proved (Bode, 1945, and Horowitz, 1963):

$$\int \log |H_{yd}(j\omega)| \, d\omega = 0. \tag{3.3}$$

Thus the area under the disturbance-rejection curve (in db-rad/sec), is *zero*. In particular, positive closed-loop disturbance rejection (i.e. $\log |H_{yd}(j\omega)| < 0$) in band implies disturbance *amplification* ($\log |H_{yd}(j\omega)| > 0$) at some frequencies out of band.

Freudenberg and Looze have recently evaluated the integral (3.3) for the SISO case, where the plant and compensator have finitely many $\mathbb{C}_+$ poles (Freudenberg and Looze, 1983). This is just Jensen's formula for $\mathbb{C}_+$ (Rudin, 1974, and Conway, 1978):

$$\int \log |H_{yd}(j\omega)| \, d\omega = 2\pi \sum_{k=1}^{K} \operatorname{Re} p_k \tag{3.4}$$

where $p_1, \ldots, p_K$ are the $\mathbb{C}_+$ poles of PC (In fact (3.4) holds when there are infinitely many $\mathbb{C}_+$ poles). Thus unstable P or C can only increase the integral (3.3): if $PC = O(s^{-2})$ then in general we have

$$\int \log |H_{yd}(j\omega)| \, d\omega \geq 0$$

regardless of whether the plant or compensator is stable or not.

Using subharmonic functions we can prove the following theorem.

MIMO BODE THEOREM. *Suppose* $PC = O(s^{-2})$. *Then*

$$\int \log \|H_{yd}(j\omega)\| \, d\omega \geq 0. \tag{3.5}$$

Proof. The hypothesis $PC = O(s^{-2})$ implies $I - H_{yd} = O(s^{-2})$. By the triangle inequality,

$$1 - \|I - H_{yd}\| \leq \|H_{yd}\| \leq 1 + \|I - H_{yd}\|$$

so that

$$\log \|H_{yd}\| = O(s^{-2}). \tag{3.6}$$

From the Poisson inequality we know $\omega \to \log \|H_{yd}(j\omega)\|$ is locally L^1; from (3.6) we know $\log \|H_{yd}(j\omega)\| = O(\omega^{-2})$; hence we conclude that

$$\omega \to \log \|H_{yd}(j\omega)\| \in L^1. \tag{3.7}$$

Now multiplying the Poisson inequality (2.5) by $\sigma_0 > 0$ and evaluating at $\omega_0 = 0$

yields

$$\frac{1}{\pi}\int \log \|H_{yd}(j\omega)\| \frac{d\omega}{1+(\omega/\sigma_0)^2} \geq \sigma_0 \log \|H_{yd}(\sigma_0)\|. \tag{3.8}$$

By (3.6), as $\sigma_0 \to \infty$ the right-hand side of (3.4) converges to zero. By dominated convergence and (3.7) the left-hand side of (3.8) converges to $\pi^{-1}\int \log \|H_{yd}(j\omega)\| \, d\omega$ and (3.5) follows. □

6 This theorem is known (John Doyle, personal communication), but we emphasize that this proof is almost the same as the proof of the SISO Bode theorem given in Freudenberg and Looze (1983), with the exception that we have used Poisson's *inequality* (valid for subharmonic functions) as opposed to Poisson's formula (valid just for *harmonic* functions). We should perhaps mention that (3.5) can be strengthened by replacing $\|H_{yd}\|$ with $\rho(H_{yd})$.

Remark. For MIMO systems *strict* inequality can occur in (3.5) even when the plant and compensator are stable (cf. SISO case (3.4)). For example consider the plant $P(s) = \mathrm{diag}\,[(s+1)^{-2}, \ (s+2)^{-2}]$ with unity compensator $C = I$. Then $\int \log \|H_{yd}(j\omega)\| \, d\omega$ is the integral of the max of two functions, each of which has integral zero by the SISO Bode theorem. Since the graphs of the two functions cross each other, $\int \log \|H_{yd}(j\omega)\| \, d\omega > 0$.

3.3 *Zames' Inequality* (Zames, 1981, and Zames and Francis, 1983)

We now consider constraints due to plant $\overline{\mathbb{C}_+}$ zeros.

ZAMES' INEQUALITY. Suppose $w \in H^\infty$ and P has a zero at $s_0 \in \overline{\mathbb{C}_+}$. Then

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega) w(j\omega)\| \geq |w(s_0)|. \tag{3.9}$$

The interpretation is as follows: usually we have $P(j\omega)C(j\omega) \to 0$ as $\omega \to \infty$ (i.e. PC is *strictly proper*), which implies that as $\omega \to \infty$, $\|H_{yd}(j\omega)\| \to 1$. Hence $\sup\{\|H_{yd}(j\omega)\| : \omega \in \mathbb{R}\} \geq 1$, that is, the minimum (unweighted) disturbance rejection is less than 0 db. The H^∞ function w in (3.9) serves to weight the disturbance-to-output map more highly in-band (where $|w|$ is large) than out-of-band (where $|w|$ is small). Zames' inequality tells us that the plant $\overline{\mathbb{C}_+}$ zero puts a lower bound on the achievable peak value of the w-weighted disturbance-to-output map.

This was proved in Zames (1981) for stable P and C and extended to unstable SISO P and C in Zames and Francis (1983).

Proof (using subharmonic functions). Under the hypotheses, $\|H_{yd}(s)w(s)\| \in \mathrm{SH}$, so invoking the maximum principle (2.3),

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega)w(j\omega)\| \geq \|H_{yd}(s_0)w(s_0)\| = \|H_{yd}(s_0)\| \, |w(s_0)| \geq |w(s_0)| \tag{3.10}$$

since by Fact 1 of §3.1, $\|H_{yd}(s_0)\| \geq 1$. □

Remark 1. We need not start with a weighting function $w(s)$ in H^∞. Indeed it is more natural simply to specify a positive weighting function $k(\omega)$ along the j

ω-axis, for example

$$k(\omega)=\begin{cases} a & \text{for } |\omega|\leq\omega_B \\ b & \text{for } |\omega|>\omega_B, \end{cases}$$

which weights the in-band disturbance rejection by $a>0$ and the out-of-band disturbance rejection by $b>0$. Using the concepts developed, it is not hard to express Zames' inequality directly in terms of the weight $k(\omega)$, as follows.

ZAMES' INEQUALITY FOR jω-AXIS WEIGHTS. Suppose $k(\omega)$ is a bounded positive function such that $\int |\log k(\omega)|\,(1+\omega^2)^{-1}\,\mathrm{d}\omega<\infty$, and P has a zero at $s_0=\sigma_0+\mathrm{j}\omega_0\in\mathbb{C}_+$. Then

$$\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)k(\omega)\|\geq\exp\left(\frac{1}{\pi}\int\log k(\omega)\,\frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+(\omega-\omega_0)^2}\right). \tag{3.11}$$

If $\sigma_0=0$ then $\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)k(\omega)\|\geq k(\omega)$.

The proof is in §A2.

This last theorem can be put in another interesting form. Suppose $M(\omega)$ is a desired upper bound for $\|H_{yd}(\mathrm{j}\omega)\|$. $M(\omega)$ would typically be small in band (to guarantee a minimum disturbance rejection), and larger, but not too large, out of band (to guarantee robustness).

COROLLARY. *Suppose that P has a zero at $\sigma_0+\mathrm{j}\omega_0\in\mathbb{C}_+$ and $M(\omega)$ is a bounded positive function such that*

$$\int\log M(\omega)\,\frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+(\omega-\omega_0)^2}<0 \tag{3.12}$$

(*this integral may be $-\infty$, but is always well defined*).

Then there is no controller C such that the closed-loop system ${}^1S(P,C)$ is stable and $\|H_{yd}(\mathrm{j}\omega)\|\leq M(\omega)$ for all $\omega\in\mathbb{R}$.

The proof is in §A2. These last two forms of Zames' inequality are related to the MIMO generalizations of the Freudenberg–Looze integral constraints which we will discuss in §3.5.

Remark 2. Zames' inequality holds for matrix weightings, i.e. if $W\in(\mathrm{H}^\infty)^{m\times k}$ and P has a zero at $s_0\in\mathbb{C}_+$, then

$$\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)\,W(\mathrm{j}\omega)\|\geq\|W(s_0)\|. \tag{3.13}$$

Note that matrix weightings do not induce symmetric seminorms (in Zames' sense). Also, the inequality is *false* if we put the matrix weighting on the left.

3.4 *MIMO Zames–Francis Inequality*

Just as the Bode integral increases when the plant or compensator is unstable (see equation (3.4)), Zames' inequality may also be sharpened when the plant or compensator is unstable. Suppose, for example, that an SISO plant P has a pole at $p_0 \in \mathbb{C}_+$ and a zero at $s_0 \in \mathbb{C}_+$, and $w \in H^\infty$. Then (Zames and Francis, 1983)

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega)w(j\omega)\| \geq \left|\frac{s_0 + \overline{p_0}}{s_0 - p_0}\right| |w(s_0)|. \tag{3.14}$$

Thus if the plant or compensator is unstable, the lower bound (3.9) increases (dramatically, if the pole p_0 and zero s_0 are close). If either the pole or zero is on the $j\omega$-axis, (3.14) is still true: it is simply Zames' inequality then.

Using subharmonic functions, we can extend this result to the MIMO case. In the MIMO case, the increase in the lower bound is not as simple as (3.14), that is, division by the Blaschke factor formed with the plant $\mathbb{C}_+$ pole: the increase depends not only on the *location* of the zero and pole in the complex plane, but also on their *geometry*, i.e. their direction in space. Let us next consider two examples.

EXAMPLE 1

$$P(s) = \frac{1}{s+1}\begin{bmatrix} \dfrac{s-0\cdot 9}{s-1} & 0 \\ 0 & \dfrac{s+0\cdot 9}{s+1} \end{bmatrix}.$$

If we apply the Zames–Francis SISO bound to channel 1 of this plant we have:

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega)w(j\omega)\| \geq 19\,|w(0.9)|.$$

In this case, the $\mathbb{C}_+$ pole-zero near cancellation has greatly reduced the achievable performance. On the other hand consider

EXAMPLE 2

$$P(s) = \frac{1}{s+1}\begin{bmatrix} \dfrac{s+0\cdot 9}{s-1} & 0 \\ 0 & \dfrac{s-0\cdot 9}{s+1} \end{bmatrix}.$$

For this plant it can be shown, using the SISO methods of Zames and Francis (1983), that for any $\varepsilon > 0$ there is a controller which yields

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega)w(j\omega)\| \leq |w(0\cdot 9)| + \varepsilon.$$

In this case the plant $\mathbb{C}_+$ pole, even though close to the plant $\mathbb{C}_+$ zero, has not degraded the performance of the system as in Example 1.

The key is that the concept of pole-zero near cancellation for MIMO systems involves the *geometry* of the pole and zero.

To make this precise:

MIMO ZAMES–FRANCIS INEQUALITY. Suppose the plant P has a pole at $p_0 \in \mathbb{C}_+$, with associated nullspace $\mathcal{N}_{\text{pole}}$ there, and a zero at $s_0 \in \mathbb{C}_+$, with associated left nullspace $\mathcal{N}_{\text{zero}}$. Then if $\|\bullet\| = \sigma_{\max}(\bullet)$,

$$\sup_{\omega\in\mathbb{R}} \|H_{yd}(j\omega)w(j\omega)\| \geq \cos\angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}}) \left|\frac{s_0+\bar{p}_0}{s_0-p_0}\right| |w(s_0)| \tag{3.15}$$

where $\cos\angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}})$ denotes the cosine of the angle between the spaces $\mathcal{N}_{\text{pole}}$ and $\mathcal{N}_{\text{zero}}$ and is defined by

$$\cos\angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}}) \triangleq \max\{|\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v}| : \boldsymbol{u}\in\mathcal{N}_{\text{zero}}, \boldsymbol{v}\in\mathcal{N}_{\text{pole}}, \|\boldsymbol{u}\| = \|\boldsymbol{v}\| = 1\}. \tag{3.16}$$

The proof is in §A2.

It is readily verified that the cosine expression in (3.15) is one in Example 1 above (indeed $\mathcal{N}_{\text{pole}} = \mathcal{N}_{\text{zero}}$ here) and zero in Example 2 above (i.e. $\mathcal{N}_{\text{pole}}$ and $\mathcal{N}_{\text{zero}}$ are orthogonal). Thus the MIMO Zames–Francis inequality tells us *nothing* for Example 2, and, more generally, is weaker than Zames' inequality when the angle between the spaces $\mathcal{N}_{\text{pole}}$ and $\mathcal{N}_{\text{zero}}$ is larger than $\arccos|(s_0-p_0)(s_0+p_0)^{-1}|$.

Remark. Although we have stated the MIMO Zames–Francis inequality for poles and zeros in the *open* right half-plane $\mathbb{C}_+$, it remains true if either the pole or zero lies on the $j\omega$-axis, since in this case the conclusion (3.15) is *weaker* than that of Zames' inequality (3.9).

3.5. *MIMO Freudenberg–Looze Integral Constraints*

Freudenberg and Looze (1983) recently applied Poisson's formula to SISO control systems to derive integral constraints similar to those appearing in Zames' inequality for $j\omega$-axis weights. Using subharmonic functions, we can extend their results to the MIMO case.

To take a simple example, suppose P and C are SISO and $P(\sigma_0) = 0$, $\sigma_0 > 0$, and $\log|H_{yd}(j\omega)| \leq -M$ for $|\omega| \leq \omega_B$ (that is, we have at least M nepers disturbance rejection up to ω_B rad/sec). Then (Freudenberg and Looze, 1983)

$$\log \sup_{\omega\in\mathbb{R}} |H_{yd}(j\omega)| \geq M\frac{\theta}{\pi-\theta}. \tag{3.17}$$

where $\theta = 2\arctan(\omega_B/\sigma_0)$.

θ can be interpreted as the total angle from the $\mathbb{C}_+$ zero σ_0 subtended by the "bandwidth" $\{j\omega : |\omega| \leq \omega_B\}$. From (3.17), we see there is quite a peak in the disturbance-to-output map H_{yd} unless $\omega_B \ll \sigma_0$.

We will now show that the same result holds for MIMO systems.

MIMO Freudenberg–Looze Constraint. Suppose that P has a zero at $\sigma_0 > 0$ and $\log\|H_{yd}(j\omega)\| \leq -M$ for $|\omega| \leq \omega_B$. Then

$$\log \sup_{\omega\in\mathbb{R}} \|H_{yd}(j\omega)\| \geq M\frac{\theta}{\pi-\theta} \tag{3.18}$$

We consider a real plant zero here for simplicity only; in fact the result holds for any $\mathbb{C}_+$ zero.

Proof. Once again the proof is nearly the same as SISO version, with the Poisson Inequality used where the Poisson formula is used in the SISO proof. From Fact 1 of §3.1, $\|H_{yd}(\sigma_0)\| \geq 1$, so $\log \|H_{yd}(\sigma_0)\| \geq 0$. From the Poisson inequality:

$$\frac{1}{\pi}\int \log \|H_{yd}(\mathrm{j}\omega)\| \frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+\omega^2} \geq \log \|H_{yd}(\sigma_0)\| \geq 0. \tag{3.19}$$

Using our hypothesis we also have

$$\frac{1}{\pi}\int \log \|H_{yd}\|(\mathrm{j}\omega)\frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+\omega^2} \tag{3.20a}$$

$$\leq \frac{-M}{\pi}\int_{-\omega_B}^{\omega_B}\frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+\omega^2} + \log\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)\|\frac{1}{\pi}\int_{|\omega|>\omega_B}\frac{\sigma_0\,\mathrm{d}\omega}{\sigma_0^2+\omega^2} \tag{3.20b}$$

$$= -M\frac{\theta}{\pi} + \log\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)\|\left(1-\frac{\theta}{\pi}\right). \tag{3.20c}$$

From (3.19) and (3.20) we conclude the expression in (3.20c) is nonnegative, and thus

$$\log\sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)\| \geq M\frac{\theta}{\pi-\theta}$$

which establishes the MIMO Freudenberg–Looze inequality. □

In fact the MIMO Freudenberg–Looze inequality can also be derived from Zames' inequality for $\mathrm{j}\omega$-axis weights. Let $R = \log\sup_{\omega\in\mathbb{R}}\|H_{yd}(j\omega)\|$ and consider the weight

$$k(\omega) = \begin{cases}\exp M & \text{for } |\omega| \leq \omega_B,\\ \exp -R & \text{for } |\omega| > \omega_B,\end{cases}$$

so that $\sup\|H_{yd}(\mathrm{j}\omega)k(\omega)\| \leq 1$. From (3.11) we have

$$1 \geq \sup_{\omega\in\mathbb{R}}\|H_{yd}(\mathrm{j}\omega)k(\omega)\| \geq \exp -(M\theta + R(\theta-\pi))$$

so that $R \geq M\theta(\pi-\theta)^{-1}$, which is (3.18).

4. Conclusion

We have given generalizations of the Paley–Wiener theorem, the Bode integral, Zames' inequality, the Zames–Francis inequality, and the Freudenberg–Looze constraints, to distributed, unstable, multiple-input multiple-output systems. We wish to emphasize the simplicity of the method. SH functions are a wide enough class to include such useful functions as $\|H_{yd}(s)\|$ and $\log\|H_{yd}(s)\|$, and yet are restricted enough to still derive meaningful constraints, e.g. via the Maximum principle or Poisson inequality.

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References

BODE, H. W. 1945 *Network Analysis and Feedback Amplifier Design*. New York: D. Van Nostrand.

CALLIER, F. M., CHAN, W. S., & DESOER, C. A. 1978 Input–output stability of interconnected systems using decompositions: An improved formulation, *Trans. Autom. Control*, vol. AC-23, no. 2, 150–163.

CALLIER, F. M., & DESOER, C. A. 1980 Stabilization, Tracking, and Disturbance Rejection in Multivariable Convolution Systems, *Annales de la Societe Scientifique de Bruxelles*, vol. 94, pp. 7–51.

CALLIER, F. M., & DESOER, C. A. 1982 *Multivariable Feedback Systems*, New York: Springer-Verlag.

CONWAY, J. B. 1978 *Functions of One Complex Variable*, New York: Springer-Verlag.

DESOER, C. A., & VIDYASAGAR, M. 1975 *Feedback Systems: Input/Output Properties*, New York: Academic Press.

DOYLE, J. C. 1982 Analysis of Feedback Systems with Structured Uncertainties, *IEE Proc.*, vol. 129, no. 6.

DOYLE, J. C., & STEIN, G. 1981 Multivariable Feedback Design: Concepts for a Classical Modern Synthesis, *IEEE Trans. Autom. Control*, vol. AC-26, no. 1, pp. 4–16.

FREUDENBERG, J. S., & LOOZE, D. P. 1983 The Impact of Right Half Plane Poles and Zeros and Design Tradeoffs in Feedback Systems, *Conf. on Decision and Control*.

HOROWITZ, I. M. 1963 *Synthesis of Feedback Systems*, New York: Academic Press.

LANG, S. 1965 *Algebra*, Reading MA: Addison-Wesley.

LOOZE, D. P., & FREUDENBERG, J. S. 1983 On the Relation Between Open Loop to Closed Loop Properties in Multivariable Feedback Systems, *Proc. ACC*, pp. 1270–1275, San Francisco, CA.

PALEY, R. E. A. C., & WEINER, N. 1934 *Fourier Transforms in the Complex Domain*, Providence, Rhode Island: American Mathematical Society.

RUDIN, W. 1974 *Real and Complex Analysis*, New York: McGraw-Hill.

ZADEH, L. A., & DESOER, C. A. 1963 *Linear System Theory: the State Space Approach*, New York: McGraw-Hill.

ZAMES, G. 1981 Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Seminorms, and Approximate Inverses, *IEEE Trans. Autom. Control*, vol. AC-26, no. 2, pp. 301–320.

ZAMES, G., & FRANCIS, B. 1983 Feedback, Minimax Sensitivity, and Optimal Robustness, *IEEE Trans. Autom. Control*, Vol. AC-28, no. 5, pp. 585–600.

Appendices

A1. Proofs of Theorems 2.1 and 2.2

We first list some elementary properties of subharmonic functions:

FACT A1. Suppose $\{f_\alpha : \alpha \in A\}$ is a family of subharmonic functions which is bounded above. Then

$$f(s) \triangleq \sup_{\alpha \in A} f_\alpha(s)$$

is subharmonic.

Proof. The function f is clearly continuous. Now suppose $\mathrm{Re}\, a > r > 0$. Then for any $\alpha \in A$,

$$f_\alpha(a) \leq \frac{1}{2\pi}\int_0^{2\pi} f_\alpha(a + re^{j\theta})\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{j\theta})\, d\theta$$

and consequently

$$f(a) = \sup_{\alpha \in A} f_\alpha(a) \leq \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{j\theta})\, d\theta,$$

which proves that f is subharmonic. □

FACT A2. Suppose f_n is a sequence of subharmonic functions and $f_n \to f$ uniformly on compact subsets of $\mathbb{C}_+$. Then f is subharmonic.

Proof. Again, f is clearly continuous. If $\operatorname{Re} a > r > 0$ then

$$\frac{1}{2\pi}\int_0^{2\pi} f(a + re^{j\theta})\, d\theta = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} f_n(a + re^{j\theta})\, d\theta \geq \lim_{n\to\infty} f_n(a) = f(a).$$

FACT A3. Suppose f is subharmonic and $\phi: [-\infty, \infty) \to [-\infty, \infty)$ is continuous, convex, and nondecreasing. Then $\phi(f(\bullet))$ is subharmonic.

Proof. Once again $\phi(f(\bullet))$ is clearly continuous, and

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(f(a + re^{j\theta}))\, d\theta \geq \phi\left(\frac{1}{2\pi}\int_0^{2\pi} f(a + re^{j\theta})\, d\theta\right) \geq \phi(f(a))$$

where the first inequality is Jensen's inequality (Rudin, 1974, p. 63) and the second follows since f is subharmonic and ϕ is non-decreasing. □

The most important case of Fact A3 is $\phi(x) = \exp x$: if f is subharmonic then $\exp f$ is subharmonic, and hence if $f \in \mathrm{SH}$ then $\exp f \in \mathrm{SH}$.

We can now prove Theorems 2.1 and 2.2.

THEOREM 2.1. *Suppose $H(s) \in (\mathrm{H}^\infty)^{m\times n}$ and suppose $\|\bullet\|$ is any induced norm. Then $\log\|H(s)\| \in \mathrm{SH}$ and $\|H(s)\| \in \mathrm{SH}$.*

Proof. In view of the remark after Fact A3, we need only prove that $\log\|H(s)\| \in \mathrm{SH}$. Property (ii) of SH is clear; property (iii) follows from (2.1) and the continuity of $A \to \log\|A\|$. It remains to show that $\log\|H(s)\|$ is subharmonic.

Continuity is clear. Let $\|\bullet\|_a$ be the norm used in $\mathbb{C}^m$ and let $\|\bullet\|_b$ be the norm *dual* to the norm used in $\mathbb{C}^n$, so that

$$\log\|H(s)\| = \sup_{\|u\|_b \leq 1, \|v\|_a \leq 1} \log|u^T H(s) v|. \qquad \text{(A1.1)}$$

For any $u \in \mathbb{C}^m$ and any $v \in \mathbb{C}^n$, the function $\log|u^T H(s) v|$ is subharmonic, so by Fact A1 and (A1.1), $\log\|H(s)\|$ is subharmonic, establishing $\log\|H(s)\| \in \mathrm{SH}$. □

Thus, for example, $\log \sigma_{\max}(H(s))$ and $\sigma_{\max}(H(s))$ are in SH.

THEOREM 2.2. *Suppose* $H(s) \in (\mathrm{H}^{\infty})^{n \times n}$. *Then*

$$\rho(H(s)) \in \mathrm{SH} \quad \textit{and} \quad \log \rho(H(s)) \in \mathrm{SH},$$

$$\mu(H(s)) \in \mathrm{SH} \quad \textit{and} \quad \log \mu(H(s)) \in \mathrm{SH}.$$

If in addition $H(s)^{-1} \in (\mathrm{H}^{\infty})^{n \times n}$, *then*

$$\mathrm{cond}_{\|}(H(s)) \in \mathrm{SH} \quad \textit{and} \quad \log \mathrm{cond}_{\|}(H(s)) \in \mathrm{SH}.$$

Proof. As in Theorem 2.1 the only hard part is showing the log-expressions are subharmonic. Suppose $H(s) \in (\mathrm{H}^{\infty})^{n \times n}$. We first show $\log \rho(H(s))$ is subharmonic. Let $\|\bullet\|$ be any induced norm, e.g. $\sigma_{\max}$. Then

$$\frac{\log \|H(s)^n\|}{n} \to \log \rho(H(s)) \quad \text{as } n \to \infty \tag{A1.2}$$

uniformly on compact subsets of $\mathbb{C}_+$. By Theorem 2.1 each $n^{-1} \log \|H(s)^n\|$ is subharmonic, so by Fact A2 and (A1.2) we conclude $\log \rho(H(s))$ is subharmonic.

If $\mu(A)$ denotes Doyle's structured singular value, then

$$\log \mu(H(s)) = \sup_{U \in \mathrm{U}'} \log \rho(UH(s))$$

where U' is the *structured unitary group* (see Doyle [Doy82]) and hence by Fact A1 and subharmonicity of $\log \rho(UH(s))$, the function $\log \mu(H(s))$ is subharmonic.

Finally suppose in addition $H(s)^{-1} \in (\mathrm{H}^{\infty})^{n \times n}$. Then $\log \mathrm{cond}_{\|}(H(s)) = \log \|H(s)\| + \log \|H(s)^{-1}\|$, and is subharmonic by Theorem 2.1. ☐

A2. Proofs of Theorems of §3.

FACT. Suppose P has a zero at $s_0 \in \mathbb{C}_+$ and a pole at $p_0 \in \mathbb{C}_+$, with associated left (right) nullspace $\mathcal{N}_{\text{zero}}$ ($\mathcal{N}_{\text{pole}}$). Then
(1) If $\boldsymbol{\lambda} \in \mathcal{N}_{\text{zero}}$ then $\boldsymbol{\lambda}^T H_{\mathbf{yd}}(s_0) = \boldsymbol{\lambda}^T$.
(2) If $\boldsymbol{\mu} \in \mathcal{N}_{\text{pole}}$ then $H_{\mathbf{yd}}(s_0)\boldsymbol{\mu} = \mathbf{0}$.

Proof. Direct calculation yields

$$H_{\mathbf{yd}} = I - N_{\mathrm{PR}} \Delta_1^{-1} N_{\mathrm{CL}} \tag{A2.1}$$

where

$$\Delta_1 \triangleq D_{\mathrm{CL}} D_{\mathrm{PR}} + N_{\mathrm{CL}} N_{\mathrm{PR}}.$$

It is shown in Callier and Desoer (1980) that the closed-loop stability Assumption 2 implies $\Delta_1^{-1} \in \mathrm{A}^{m \times m}$. In fact $\det \Delta_1$ is what is usually called the *characteristic function* of the closed-loop system ${}^1S(P, C)$, and the closed-loop stability assumption (Assumption (2)) is equivalent to $(\det \Delta_1)^{-1} \in \mathrm{A}$. If $\boldsymbol{\lambda} \in \mathcal{N}_{\text{zero}}$ then $\boldsymbol{\lambda}^T N_{\mathrm{PR}}(s_0) = 0$, hence from (A2.1) $\boldsymbol{\lambda}^T H_{\mathbf{yd}}(s_0) = \boldsymbol{\lambda}^T$, which establishes Fact (1). To prove Fact (2), we note that

$$H_{\mathbf{yd}} = D_{\mathrm{CR}} \Delta_2^{-1} D_{\mathrm{PL}} \tag{A2.2}$$

where

$$\Delta_2 \triangleq D_{\mathrm{PL}} D_{\mathrm{CR}} + N_{\mathrm{PL}} N_{\mathrm{CR}}$$

and, as above, $\Delta_2^{-1} \in \mathrm{A}^{m\times m}$. If $\boldsymbol{\mu} \in \mathcal{N}_{\text{pole}}$ then $D_{pL}(p_0)\boldsymbol{\mu} = \mathbf{0}$, so from (A2.2) we conclude $H_{yd}(p_0)\boldsymbol{\mu} = \mathbf{0}$. □

ZAMES' INEQUALITY FOR jω-AXIS WEIGHTS. Suppose $k(\omega)$ is a bounded positive function such that $\int |\log k(\omega)|\,(1+\omega^2)^{-1}\,d\omega < \infty$, and P has a zero at $s_0 = \sigma_0 + j\omega_0 \in \mathbb{C}_+$. Then

$$\sup_{\omega\in\mathbb{R}} \|H_{yd}(j\omega)k(\omega)\| \geq \exp \frac{1}{\pi} \int \log k(\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2}. \tag{A2.3}$$

Remark. The hypothesis on k is precisely the condition that there exists $w \in \mathrm{H}^\infty$
7 with $k(\omega) = |w(j\omega)|$. We will not directly use this fact.

Proof. We first extend $\log k(\omega)$ to a function h harmonic in $\mathbb{C}_+$. Define h by $h(j\omega) \triangleq \log k(\omega)$ and for $\sigma_0 > 0$,

$$h(\sigma_0 + j\omega_0) \triangleq \frac{1}{\pi} \int \log k(\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2}. \tag{A2.4}$$

The hypothesis on k ensures that the integral in (A2.4) makes sense. It can be directly verified that $h \in \mathrm{SH}$ (indeed h is harmonic, i.e. $-h \in \mathrm{SH}$ as well). Hence

$$\log \|H_{yd}(s)\| + h(s) \in \mathrm{SH}$$

so by Fact A3 of §A1,

$$\|H_{yd}(s)\| \exp h(s) \in \mathrm{SH}.$$

By the Poisson inequality,

$$\sup_{\omega\in\mathbb{R}} \|H_{yd}(j\omega)k(\omega)\| \geq \|H_{yd}(s_0)\| \exp h(s_0)$$

$$\geq \exp \frac{1}{\pi} \int \log k(\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2}$$

since $\|H_{yd}(s_0)\| \geq 1$ by Fact (1) of §3.1. □

COROLLARY. *Suppose that P has a zero at $\sigma_0 + j\omega_0 \in \mathbb{C}_+$ and $M(\omega)$ is a bounded positive function such that*

$$\int \log M(\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2} < 0. \tag{A2.5}$$

Then there is no controller C such that the closed loop system ${}^1S(P, C)$ is stable and $\|H_{yd}(j\omega)\| \leq M(\omega)$ for all $\omega \in \mathbb{R}$.

Proof. By contradiction. Suppose there is such a controller. Then

$$\frac{1}{\pi} \int \log \|H_{yd}(j\omega)\| \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2} \leq \frac{1}{\pi} \int \log M(\omega) \frac{\sigma_0\,d\omega}{\sigma_0^2 + (\omega-\omega_0)^2} < 0. \tag{A2.6}$$

Since P has a zero at $s_0 = \sigma_0 + j\omega_0 \in \mathbb{C}_+$, we have as above

$$0 \leqslant \log \|H_{yd}(s_0)\| \leqslant \frac{1}{\pi} \int \log \|H_{yd}(j\omega)\| \frac{\sigma_0 \, d\omega}{\sigma_0^2 + (\omega - \omega_0)^2} \tag{A2.7}$$

which contradicts (A2.6), establishing the Corollary. □

MIMO Zames–Francis Inequality. Suppose P has a pole at $p_0 \in \mathbb{C}_+$ and a zero at $s_0 \in \mathbb{C}_+$, with associated left (right) nullspace $\mathcal{N}_{\text{pole}}$ ($\mathcal{N}_{\text{zero}}$), respectively. Then if $\|\bullet\| = \sigma_{\max}(\bullet)$,

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega) w(j\omega)\| \geqslant (\cos \angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}})) \left| \frac{s_0 + p_0}{s_0 - p_0} \right| |w(s_0)| \tag{A2.8}$$

where $\cos \angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}})$ is defined in (3.16) of §3.3.

Proof. Let U_{pole} be a matrix whose columns are an orthonormal basis for $\mathcal{N}_{\text{pole}}$. By Fact (1) of §3.1, $H_{yd}(p_0) U_{\text{pole}} = 0$. Since H_{yd} is analytic at p_0, we have

$$\frac{s + \overline{p_0}}{s - p_0} H_{yd}(s) U_{\text{pole}} \in (\mathrm{H}^\infty)^{n \times \dim \mathcal{N}_{\text{pole}}}. \tag{A2.9}$$

Now since $\|U_{\text{pole}}\| \leqslant 1$ (recall $\|\bullet\| = \sigma_{\max}(\bullet)$ here!),

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega) w(j\omega)\| \geqslant \sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega) U_{\text{pole}} w(j\omega)\| = \sup_{\omega \in \mathbb{R}} \left\| \frac{j\omega + \overline{p_0}}{j\omega - p_0} H_{yd}(j\omega) U_{\text{pole}} w(j\omega) \right\|$$

and using (A2.9) and the Poisson inequality:

$$\geqslant \left\| \frac{s_0 + \overline{p_0}}{s_0 - p_0} H_{yd}(s_0) U_{\text{pole}} w(s_0) \right\| = \left| \frac{s_0 + \overline{p_0}}{s_0 - p_0} \right| |w(s_0)| \, \|H_{yd}(s_0) U_{\text{pole}}\|. \tag{A2.10}$$

Now let U_{zero} be a matrix whose columns form an orthonormal basis for $\mathcal{N}_{\text{zero}}$. Then by Fact (2) of §3.1 we have $U_{\text{zero}}^{\mathrm{T}} H_{yd}(s_0) = U_{\text{zero}}^{\mathrm{T}}$. Since $\|U_{\text{zero}}^{\mathrm{T}}\| \leqslant 1$,

$$\|H_{yd}(s_0) U_{\text{pole}}\| \geqslant \|U_{\text{zero}}^{\mathrm{T}} H_{yd}(s_0) U_{\text{pole}}\| = \|U_{\text{zero}}^{\mathrm{T}} U_{\text{pole}}\| \tag{A2.11a}$$

$$= \max \{ |u^{\mathrm{T}} v| : u \in \mathcal{N}_{\text{zero}}, v \in \mathcal{N}_{\text{pole}}, \|u\| = \|v\| = 1 \} \tag{A2.11b}$$

$$\triangleq \cos \angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}}). \tag{A2.11c}$$

From (A2.10) and (A2.11) we conclude

$$\sup_{\omega \in \mathbb{R}} \|H_{yd}(j\omega) w(j\omega)\| \geqslant \cos \angle(\mathcal{N}_{\text{pole}}, \mathcal{N}_{\text{zero}}) \left| \frac{s_0 + \overline{p_0}}{s_0 - p_0} \right| |w(s_0)|$$

which establishes the MIMO Zames–Francis inequality. □