# Uniqueness of a Basic Nonlinear Structure

STEPHEN BOYD AND LEON O. CHUA, FELLOW, IEEE

*Abstract*—**In this paper we show that systems consisting of a memoryless nonlinearity sandwiched between two linear time-invariant (LTI) operators are *unique* modulo scaling and delays. We mention a few corollaries and applications of general circuit and system theoretic interest.**

## I. Introduction

In nonlinear systems theory two types of operators are especially important: *linear time-invariant* (LTI) operators and *memoryless* or *static* nonlinear operators. Many important and well-known results pertain to systems which are interconnections of these operators, for example, the Popov criterion for the Lur'e structure. Indeed if multi-input multi-output (MIMO) operators are considered, all dynamical systems are included.

In this paper we consider what is perhaps the simplest interconnection of these operators, shown in Fig. 1, and ask the question: in what sense are such systems unique, that is, under what conditions could two such systems have the same input–output (I/O) map? Some conditions are easy to think of, for example, we can rescale the operators or distribute any delay in $A$ and $C$ arbitrarily between them ($\tilde{A} = \alpha\exp(-sT)A$, $\tilde{C} = \gamma\exp(sT)C$, $\tilde{B}(x) = \alpha^{-1}B(\gamma^{-1}x)$). We show that these are the *only* ways these systems fail to be unique.

Rugh and others [1]–[5] have shown that certain systems containing lumped LTI operators and memoryless power nonlinearities or multipliers are unique in a certain sense, and this paper is inspired by their work. Our emphasis, however, is slightly different: we consider memoryless nonlinearities as opposed to multipliers and pure power nonlinearities, and general as opposed to lumped LTI operators.

## II. Notation

We consider operators with a Volterra series:

$$Nu(t) = \sum_{n=1}^{\infty} y_n(t)$$

$$y_n(t) = \int\cdots\int h_n(\tau_1,\tau_2,\cdots\tau_n)u(t-\tau_1)u(t-\tau_2)\cdots u(t-\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n$$

where $h_n$ is a symmetric real tempered distribution supported on $(R^+)^n$, and the inputs $u$ belong to some subset

Manuscript received December 20, 1982, revised February 9, 1983. This work was supported in part by the Office of Naval Research under Contract N00014-76-C-0572, the National Science Foundation under Grant ECS 80-20-640, and the Fannie and John Hertz Foundation.
The authors are with the Department of Electrical Engineering and Computer Sciences, and the Electronics Research Laboratory, University of California, Berkeley, CA, 94720.

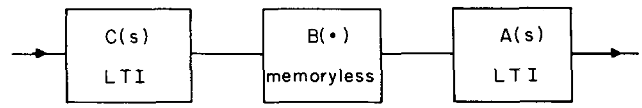


Fig. 1. The system considered: $A$ and $C$ are LTI with frequency responses $A(s)$ and $C(s)$, $B$ is memoryless with characteristic function $B(x)$.

of $C^\infty(R^+)$ which ensures $y_n$ summable.[1] We refer to $h_n$ as the $n$th time-domain Volterra kernel of $N$; we will work with their Laplace transforms, called the (frequency-domain) Volterra kernels or nonlinear transfer functions:

$$H_n(s_1,s_2,\cdots s_n) = \int\cdots\int h_n(t_1,t_2,\cdots t_n)\cdot\exp\left(-\sum_{i=1}^{n} s_i t_i\right)dt_1\,dt_2\cdots dt_n$$

defined and analytic at least in $\{s|\mathrm{Re}\,s_k>0,\ k=1\cdots n\}$, henceforth denoted $(C^+)^n$. For more details, see [6]–[11].

A LTI operator has all kernels above the first vanishing; a memoryless operator is one with all kernels constant, and a positive radius of convergence. To keep the notation simple, we will use the same symbol for a LTI operator and its first kernel, and similarly for a memoryless operator and its associated function from $R$ to $R$. Juxtaposition of operators will denote composition, equality of operators will mean that they have the same I/O map.

We should mention that the Volterra kernels are completely determined by the operator $N$, i.e., by its I/O map. Indeed for $\omega_k$ distinct and nonzero,

$$H_n(j\omega_1,\cdots j\omega_n) = \frac{1}{n!}\frac{\partial^n}{\partial\alpha_1\cdots\partial\alpha_n}\left[\overline{N\left(2\sum_{k=1}^{n}\alpha_k\cos\omega_k t\right)}\right]\left(\sum_{k=1}^{n} j\omega_k\right)\Bigg|_{\alpha=0}$$

where the right-hand side refers only to the *operator* $N$, and not to any particular representation of it. This means that the kernels can be measured [12].

## III. Statement and Proof of Theorem

*Theorem 1:* Suppose $A$, $\tilde{A}$, $C$, $\tilde{C}$ are nonzero LTI operators, $B$ and $\tilde{B}$ are memoryless operators, at least one of which is not linear. If $ABC = \tilde{A}\tilde{B}\tilde{C}$, then there are real constants $\alpha$, $\gamma$, $T$ such that

$$\tilde{A}(s) = \alpha\exp(-sT)A(s)$$

[1]This formulation includes operators such as differentiation and has a correspondingly restricted signal space. If you like, the $h_n$ can be bounded measures, the signal space the open ball in $L^\infty$ with radius $R^{-1} = \limsup\|h_n\|^{1/n}$.

$$\tilde{C}(s) = \gamma \exp(sT) C(s)$$
$$\tilde{B}(x) = \alpha^{-1} B(\gamma^{-1} x).$$

That is, systems of the form (1) which are not linear have a unique representation of the form (1), modulo scaling, and delays.

*Proof:* Under the hypotheses of Theorem 1, the two systems have the same kernels

$$H_n = A(s_1 + \cdots + s_n) B_n C(s_1) C(s_2) \cdots C(s_n) \quad (1)$$
$$= \tilde{A}(s_1 + \cdots + s_n) \tilde{B}_n \tilde{C}(s_1) \tilde{C}(s_2) \cdots \tilde{C}(s_n). \quad (2)$$

Consider now any $n > 1$ for which $H_n$ is not identically zero (and there is at least one such $n$). Find an open ball $D$ in $(C^+)^n$ on which $H_n \neq 0$. Indeed $\{s \in (C^+)^n | H_n(s) \neq 0\}$ is open and connected in $(C^+)^n$. On $D$ define

$$Q = \ln\left[B_n (C/\tilde{C})(s_1) \cdots (C/\tilde{C})(s_n)\right] \quad (3)$$
$$= \ln\left[\tilde{B}_n (\tilde{A}/A)(s_1 + \cdots + s_n)\right]. \quad (4)$$

Any branch of ln will do. Then on $D$,

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = 0 \quad (5)$$

when calculated from (3) and

$$\frac{\partial^2 Q}{\partial s_1 \partial s_2} = \left[\ln(\tilde{A}/A)\right]''(s_1 + \cdots + s_n) \quad (6)$$

when calculated from (4). Note that $n > 1$ is *crucial*; this is where the requirement of *strict* nonlinearity enters. From (5) and (6) we conclude for some $\eta$ and $T$

$$\ln(\tilde{A}/A)(s_1 + \cdots + s_n) = \eta - T(s_1 + \cdots + s_n) \quad (7)$$

on $D$ and hence *everywhere* in $(C^+)^n$. Thus

$$\tilde{A}(s) = \alpha \exp(-sT) A(s) \quad (8)$$

for $s \in C^+$, where $\alpha = \exp \eta$. From $A(\bar{s}) = \overline{A(s)}$ we conclude $\alpha$ and $T$ are real. Substituting (8) into (1) and (2) yields

$$\tilde{C}(s) = \gamma \exp(sT) C(s) \quad (9)$$

where $\gamma^n = B_n \tilde{B}_n^{-1} \alpha^{-1}$ and as above $\gamma$ real. Thus we have $\tilde{B}_n = \alpha^{-1} B_n \gamma^{-n}$, which remains true for those $n$ for which $B_n = \tilde{B}_n = 0$, hence

$$\tilde{B}(x) = \alpha^{-1} B(\gamma^{-1} x) \quad (10)$$

and the theorem is proved.

*Corollary 1:* Systems of the form $HN$ are *completely disjoint* from systems of the form $NH$, where $H$ is LTI *nonconstant* and $N$ is memoryless *strictly* nonlinear. (See Fig. 2.) This may be obvious for other reasons, for example, if $h$ is absolutely continuous then the kernels of $HN$ are absolutely continuous whereas those of $NH$ are singular measures.

*Corollary 2:* Given any operator $N$ with at least two nonzero kernels, the only LTI operators which commute with $N$ are delays (or delays and negation, if $N$ is odd).

*Corollary 3:* Chua [13] has defined *algebraic* circuit elements as those with constitutive relations of the form $\Phi(v^{(\alpha)}, i^{(\beta)}) = 0$ (where $f^{(\alpha)}$ is the $\alpha$th derivative, or integral if $\alpha < 0$, of $f$). Nonlinear resistors, capacitors, and

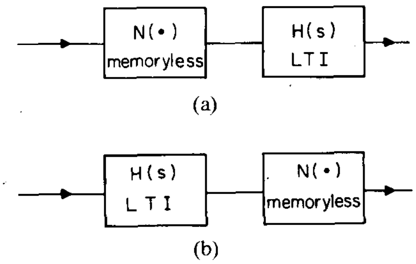


Fig. 2. Two simple nonlinear structures: (a) is of the form $HN$, and (b) is of the form $NH$. Except in the trivial cases $H$ constant or $N$ linear, the two types are exclusive.

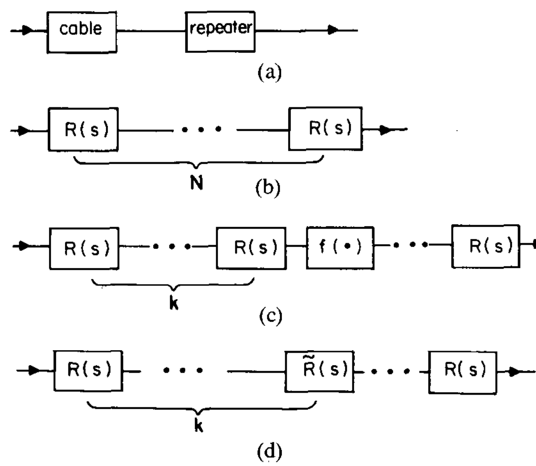


Fig. 3(a). Cable-repeater section. (b) Communications system. (c) System with $k$th repeater nonlinearly faulted at output. (d) System with $k$th repeater linearly faulted.

inductors are examples. Under weak conditions Theorem 1 shows that if such an element is *strictly nonlinear* its order $(\alpha, \beta)$ and its characteristic curve $\Phi(x, y) = 0$ are *unique*, that is, such elements have only one description as algebraic elements.

*Application:* Consider a communications system consisting of $N$ cable-repeater sections, each with frequency response $R(s)$. Suppose the output stage of the $k$th repeater drifts off bias and starts distorting slightly. The faulted system I/O operator is then $R^{N-k} f(\cdot) R^k$, where $f(\cdot)$ represents the nonlinear output stage: see Fig. 3. Theorem 1 tells us that *from I/O measurements alone* (of the whole system) we can locate the faulty repeater.[2] This should be compared to a linear fault: suppose an element in the $k$th repeater amplifier drifts in such a way as to, say, halve the bandwidth of the repeater. The $k$th repeater is still linear, but with frequency response $\tilde{R}(s)$. I/O measurements alone *cannot locate* this fault, since the system's linear (and only) frequency response is $R(s)^{N-1} \tilde{R}(s)$ no matter where the fault is.

## IV. Comments and Generalizations

The theorem remains true under a wide variety of generalizations. It is true for discrete-time systems, with the obvious modification of the conclusion $\tilde{A}(z) = \alpha z^{-d} A(z)$ and $\tilde{C}(z) = \gamma z^d C(z)$, $d$ an integer. It holds for multidimensional systems as well, for example, for two-dimensional

[2] One might suspect that this is possible. The advantage of our machinery is that it can tell us exactly which distortion products to look at.

systems we get

$$H_n(s_1,\cdots s_n;p_1,\cdots p_n)=A(s_1+\cdots+s_n;p_1+\cdots+p_n)\cdot B_nC(s_1,p_1)\cdots C(s_n,p_n)$$

and a proof analogous to the one above establishes

$$\tilde{A}(s,p)=\alpha\exp(-sX-pY)A(s,p)$$
$$\tilde{C}(s,p)=\gamma\exp(sX+pY)C(s,p)$$
$$\tilde{B}(x)=\alpha^{-1}B(\gamma^{-1}x).$$

The theorem is also true for most noncausal $A$ and $C$. For example, when their impulse responses fall off exponentially $A(s)$ and $C(s)$ are analytic in a strip $-\epsilon<\mathrm{Re}\,s<\epsilon$ and the proof above applies directly. And under weaker conditions it is usually true as a consequence of the fact that the functional equation $f(x+y)=\alpha g(x)g(y)$ only has exponential solutions under quite general conditions, e.g., $f$ and $g$ measurable and nonzero.[3] But there are pathological cases in which the theorem fails, for example, consider

$$A(j\omega)=\begin{cases}1 & |\omega|<1\\ 0 & |\omega|\geqslant 1\end{cases}$$
$$B(x)=x^2$$
$$C(j\omega)=\begin{cases}1 & |\omega|<3\\ 0 & |\omega|\geqslant 3.\end{cases}$$

Then $ABC=ABI$, where $I(s)=1$.

From these comments we may conclude, for example, that the theorem holds for image processing operators of the form (1). Other generalizations, however, are not straightforward. We do not know under what conditions the theorem holds in the MIMO case. We suspect but cannot prove that the theorem holds for any measurable nonlinearity, and not just the analytic ones considered here.

We have recently shown [15] that systems containing *one* SISO memoryless nonlinearity (possibly in a feedback loop) are unique modulo scaling, delays, and loop transformation. This result applies directly to circuits containing one nonlinear element. The argument is slightly more involved and can be found in [15].

## V. A Stable Decomposition Method

Our proof, which relies on partial derivatives and analytic continuation, might suggest that the decomposition of $H_n$ into $A(s)$, $B_n$, and $C(s)$ is quite sensitive. The main purpose of this section is to show that this is not so. We now give a sketch of the simplest case: discrete time, minimum phase exponentially decaying $A$ and $C$. We decompose the second kernel since the higher order kernels decompose similarly. We assume that $H_2$ has been measured: there are simpler methods to estimate $A$ and $C$ based on partial knowledge of $H_2$ (e.g., from $H_2(e^{j\Omega},e^{-j\Omega})=A(0)B_2|C(e^{j\Omega})|^2$; cf. [2], [3]) but measuring the kernels allows us to *verify* that the system has the form (1), as well as estimate $A$ and $C$. It will be convenient to normalize $A(0)=C(0)=1$. Then $\ln H_2$ is analytic in $\{(z_1,z_2)||z_1|\leqslant 1, |z_2|\leqslant 1\}$ and

$$\ln H_2(e^{j\Omega_1},e^{j\Omega_2})=\ln A(e^{j(\Omega_1+\Omega_2)})+\ln B_2+\ln C(e^{j\Omega_1})+\ln C(e^{j\Omega_2}).$$

The assumptions imply that the terms above containing $A$, $B_2$, and $C$, when considered elements of $L_2(T\times T)$,[4] are contained in the mutually orthogonal subspaces $S_1$, $S_2$, and $S_3$, where

$$S_1=\left\{g(\Theta_1+\Theta_2)|g\in L_2(T),\ \int g=0\right\}$$
$$S_3=\left\{f(\Theta_1)+f(\Theta_2)|f\in L_2(T),\ \int f=0\right\}$$

and $S_2$ is the constants. A natural method to estimate $\ln A$, $\ln B_2$, and $\ln C$ is to project $\ln H_2$ on these subspaces, i.e.

$$\ln B=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j\Omega_1},e^{j\Omega_2})\,d\Omega_1\,d\Omega_2$$
$$\ln A(e^{j\Omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j(\Omega-\Omega_1)},e^{j\Omega_1})\,d\Omega_1-\ln B$$
$$\ln C(e^{j\Omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln H_2(e^{j\Omega},e^{j\Omega_1})\,d\Omega_1-\ln B.$$

In fact these formulas can be used to estimate $\ln|A|$, $\ln|C|$, and $B_2$ when $A$ or $C$ is not minimum phase,[5] but the method must be modified to yield the correct phases. The point is that $A$, $B_2$, and $C$ can be estimated in a stable way, without taking partial derivatives.

## VI. Conclusion

The theorem has the interpretation that from I/O measurements alone, we can in principle extract information about the *internal structure* of a system of the form (1). We believe that this is an instance of a general property of nonlinear systems: the same complexity which makes nonlinear systems difficult to represent, analyze, and design (e.g., noncommutativity, nondistributivity $\cdots$) also allows much more information about internal structure to be extracted from I/O measurements.

## References

[1] W. W. Smith and W. J. Rugh, "On the structure of a class of nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 701–706, Dec. 1974.
[2] S. L. Baumgartner and W. J. Rugh, "Complete identification of a class of nonlinear systems from steady state frequency response," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 753–759, Sept. 1975.
[3] E. M. Wysocki and W. J. Rugh, "Further results on the identification problem for the class of nonlinear systems $S_M$," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 664–670, Nov. 1976.
[4] K. S. Shanmugam and M. Lal, "Analysis and synthesis of a class of nonlinear systems," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 17–25, Jan. 1976.
[5] T. R. Harper and W. J. Rugh, "Structural features of factorable Volterra systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 822–832, Dec. 1976.
[6] I. W. Sandberg, "Expansions for nonlinear systems," *Bell Syst. Tech. J.*, vol. 61, pp. 159–200, Feb. 1982.

[3] See, e.g., Shapiro [14].

[4] $T$ is the unit circle with normalized measure.

[5] With the modified normalization for the transfer functions $A(0)=\Pi_k|\alpha_k|$, where the $\alpha_k$ are the zeros of $A$ in the unit disk.

[7] R. deFigueiredo, "A generalized Fock space framework for nonlinear system and signal analysis," *IEEE Trans. Circuits Syst.*, pp. 635–645, this issue.
[8] D. D. Weiner and J. Spina, *Sinusoidal Analysis and Modeling of Weakly Nonlinear Circuits*. New York: Van Nostrand, 1980.
[9] W. J. Rugh, *Nonlinear System Theory: The Volterra/Wiener Approach*. Baltimore, MD: The Johns Hopkins Univ. Press, 1981.
[10] L. O. Chua and C. Y. Ng, "Frequency domain analysis of nonlinear systems: general theory," *IEE J. Elect. Circuits Syst.*, vol. 3, no. 4, pp. 165–185.
[11] L. O. Chua and C. Y. Ng, "Frequency domain analysis of nonlinear systems: formulation of transfer functions," *IEE J. Electr. Circuits Syst.*, vol. 3, no. 6, p. 257–267, Nov. 1979.
[12] S. Boyd, Y. S. Tang, and L. O. Chua, "Measuring Volterra kernels," *IEEE Trans. Circuits Syst.*, vol. CAS-30, pp. 571–577, Aug. 1983.
[13] L. O. Chua, "Device modeling via basic nonlinear circuit elements," *IEEE Trans. Circuits Syst.*, vol. CAS-27, pp. 1014–1044, Nov. 1980.
[14] H. N. Shapiro, "*A micronote on a linear functional equation*," *Amer. Math. Monthly*, vol. 80, pp. 1041, Nov. 1973.
[15] S. Boyd and L. O. Chua, "Uniqueness of systems and circuits containing one nonlinearity nonlinearity," *Mathematical Theory of Networks and Systems Conf. Proc.*, Ber Sheba, Israel, Springer Verlag, 1984, to be published.

**Stephen Boyd** received the A.B. degree in mathematics (summa cum laude) from Harvard University, Cambridge, MA, in 1980. He then entered the mathematics Ph.D. program at the University of California, Berkeley, and in 1981 transferred to electrical engineering.

In graduate school he has held the following fellowships: University of California Regents Fellowship (Mathematics), National Science Foundations Graduate Fellowship (Mathematics), Irving and Lucille Smith Fellowship (EECS), and is now a Fannie and John Hertz Fellow. He is interested in nonlinear systems and control theory.

**Leon O. Chua** (S'60–M'62—SM'70–F'74), for a photograph and biography please see page 617 of this issue.

# The Effects of Small Noise on Implicitly Defined Nonlinear Dynamical Systems

S. SHANKAR SASTRY, MEMBER, IEEE

*Abstract* —**The dynamics of a large class of nonlinear systems are described implicitly, i.e., as a combination of algebraic and differential equations. These dynamics admit of jump behavior. We extend the deterministic theory to a stochastic theory since (i) the deterministic theory is restrictive, (ii) the macroscopic deterministic description of dynamics frequently arises from an aggregation of microscopically fluctuating dynamics, and (iii) to robustify the deterministic theory. We compare the stochastic theory with the deterministic one in the limit that the intensity of the additive white noise tends to zero. We study the modeling issues involved in applying this stochastic theory to the study of the noise behavior of a multivibrator circuit, discuss the limitations of our methodology for certain classes of systems and present a modified approach for the analysis of sample functions of noisy nonlinear circuits.**

## I. Introduction

THE DYNAMICS of a large class of engineering systems are described only implicitly, for instance, those of nonlinear circuits, swing dynamics of an interconnected power system, as also thermodynamic systems far from equilibrium. The implicit definition of their dynamics is as follows: the state variables are constrained to satisfy some algebraic equations, i.e., they are constrained to lie on a manifold $M$ in the state space. The dynamics on this manifold $M$ are then specified implicitly by specifying only the projection of the vector field on $M$ onto a certain base space above which $M$ lies. (i.e., a subspace of the original state space of the same dimension as $M$). The process of obtaining the system dynamics explicitly consists of "lifting" the specified velocities onto a vector field on $M$ (lifting is the inverse of projecting). Lifting may not, however, be possible at points where the projection map (restricted to the tangent space of the constraint manifold) has singularities. This singularity is typically resolved by *regularization*, i.e., by interpreting the algebraic constraint equations as the singularly perturbed limit of "parasitic" or fast dynamics. The dynamics of the original system are obtained as the degenerate limit of the dynamics of the regularized system—the resulting trajectories may be discontinuous and this is referred to as *jump behavior*.

The foregoing deterministic theory needs to be extended to a stochastic theory for three reasons:

a) The conditions under which the limit trajectories to the regularizations exist are extremely restrictive so as to exclude several systems of interest.

Manuscript received October 12, 1982; revised January 25, 1983. This work was supported in part by the Air Force Office of Scientific Research under Grant AFOSR-82-0258.
The author was with the Laboratory for Information and Decision Systems, M.I.T., Cambridge, MA 02139. He is now with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720.

0098-4094/83/0900-0651$01.00 ©1983 IEEE