Volterra Series for Nonlinear Circuits

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When is the Volterra series representation

$$y(t) = h_0 + \sum_{n=1}^{\infty} \int \ldots \int h_n(\tau_1, \ldots, \tau_n) u(t-\tau_1) \ldots u(t-\tau_n) d\tau_1 \ldots d\tau_n \quad (1)$$

appropriate for a nonlinear circuit? In (1), u might be the port current and y the port voltage of a nonlinear one-port, so that (1) is a representation of the nonlinear *impedance operator* of the one-port.

The answer to this question depends on what we mean by *appropriate*. If we ask, for what circuits is there an *exact* relation of the form (1), for u small enough, the definitive answer comes from the recent work of Sandberg.[1, 2, 3] Roughly speaking, the requirement is that the the constitutive relations of the various elements be *analytic*, that is, have a power series which converges at least for small deviations from the bias point, and that the linearized network be exponentially stable.

Thus the question of exact representation by the Volterra series (1) is answered, but the answer is not really in terms of direct engineering significance. We would not ask a technician to check whether the *V-I* curve of some device is analytic, as opposed to merely infinitely differentiable, since all modeling of *real circuits* applies only within a certain *precision*, for a certain set of input signals. Thus we are led to a different question: when can, say, the impedance operator $\mathbf{Z}$ of a nonlinear circuit be approximated within some precision ϵ over some useful set K of signals by a Volterra series operator $\mathbf{Z}_{vlt}$, that is

$$||\mathbf{Z}u - \mathbf{Z}_{vlt}|| \leq \epsilon$$

for all $u \in K$?

The answer is simply that $\mathbf{Z}$ must have *fading memory*, which makes good engineering sense. Thus we may answer our (modified) question:

> *The Volterra series representation (1) is appropriate for circuits with fading memory.*

One important comment: while the class of circuits with fading memory is a very wide class of circuits, it excludes those circuits which exhibit truly nonlinear behavior. To give one important example, circuits with several stable equilibria do not have fading memory for large inputs, but if the input level is small enough, they generally do.

References

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